ASYMPTOTICALLY CONICAL CALABI-YAU MANIFOLDS, I

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Abstract. This is the first part in a two-part series on complete Calabi-Yau manifolds asymptotic to Riemannian cones at infinity. We begin by proving general existence and uniqueness results. The uniqueness part relaxes the decay condition $O(r^{-n-\varepsilon})$ needed in earlier work to $O(r^{-\varepsilon})$, relying on some new ideas about harmonic functions. We then look at a few examples: (1) Crepant resolutions of cones. This includes a new class of Ricci-flat small resolutions associated with flag manifolds. (2) Affine deformations of cones. One focus here is the question of the precise rate of decay of the metric to its tangent cone. We prove that the optimal rate for the Stenzel metric on T^*S^n is $-2\frac{n}{n-1}$.

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1. Introduction

1.1. **Background.** Consider a complete Riemannian manifold (M, g) with $\mathrm{Ric} \geq 0$ and Euclidean volume growth. Cheeger-Colding theory [15] implies that each blowdown sequence $(M, \lambda_i g)$, $\lambda_i \to 0$, subconverges in the pointed Gromov-Hausdorff sense to the cone over some length space. If $\mathrm{Ric} = 0$, then this cone is likely to be the same for all sequences, so that M would be *asymptotically conical* in a Gromov-Hausdorff sense. This was proved by Cheeger and Tian [16], assuming $|\mathrm{Rm}| = O(r^{-2})$ (so that the links of all bona fide tangent cones are *smooth*) and an integrability condition. In the course of their proof, they in fact establish C^∞ convergence to one such cone at a rate of $O(r^{-\varepsilon})$.

In this paper and its sequel, we wish to construct a careful theory of complete Ricci-flat Kähler manifolds that are asymptotically conical ("AC") in this stronger sense. If we think of noncompact Calabi-Yau's as bubbles in the singularity formation of compact Kähler-Einstein manifolds, then at least heuristically, these are precisely the ones that correspond to noncollapsed isolated singularities. Besides [16], the foundational paper in this area is Tian-Yau [50], but see also [23, 33, 53].

Remark 1.1. Ricci-flat Kähler manifolds of Euclidean volume growth whose tangent cones are *not* smooth, and in fact not even products of smooth cones, exist as well: Joyce's QALE spaces [33] are desingularizations of very general flat orbifolds $\mathbb{C}^n/\Gamma$. Biquard-Gauduchon [6] wrote down explicit hyper-Kähler examples whose tangent cones are realized as nilpotent orbit closures in $\mathfrak{sl}(N, \mathbb{C})$.

The following three basic AC Ricci-flat examples are helpful to keep in mind throughout.

Example 1.2. Calabi's metric on the total space of $K_{\mathbb{P}^{n-1}}$ [11], whose tangent cone is $\mathbb{C}^n/\mathbb{Z}_n$. The underlying complex manifold is a *crepant resolution* of the Calabi-Yau cone $\mathbb{C}^n/\mathbb{Z}_n$.

Example 1.3. Stenzel's metric on T^*S^n [49]. Holomorphically, this is a *deformation* of the ordinary double point in $\mathbb{C}^{n+1}$, containing a special Lagrangian S^n rather than a holomorphic $\mathbb{P}^{n-1}$.

These two examples are related by a hyper-Kähler rotation if $n = 2$. If $n > 2$, then the first cone has no deformations. If $n > 3$, then the second cone has no crepant resolutions. Both manifolds are quasiprojective and may be compactified by adding positive divisors at infinity.

Example 1.4. Construct $M = X \setminus D$, where X is a singular cubic 3-fold $z_1F(z) + z_2G(z) = 0$ in $\mathbb{P}^4$ blown up along the plane $z_1 = z_2 = 0$, and D denotes the proper transform of a hyperplane section avoiding the 4 singularities. Then M has a maximal compact analytic subset consisting of 4 copies of $\mathbb{P}^1$. AC Ricci-flat metrics on M were obtained in [50]. This example represents a *bubble tree* from the point of view of singularity formation: The tangent cone to the singularity is the complex cone over a cubic surface; the deepest bubbles are small resolutions of the conifold [12].

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1.2. **Results.** The theoretical part of this paper (Sections 2–3) can be summarized as follows.

Existence and uniqueness. *Let M be a complex manifold of dimension $n > 2$ with a holomorphic volume form Ω, and C a Calabi-Yau cone with radius function r and holomorphic volume form Ω_0. Let Φ be a diffeomorphism from C to M at infinity such that $\Phi^*\Omega - \Omega_0 = O(r^{-\varepsilon})$ for some $\varepsilon > 0$. If $\mathfrak{k} \in H^2(M)$ contains a Kähler form satisfying a mild technical condition, then for each $c > 0$, $\mathfrak{k}$ also contains a unique Ricci-flat Kähler form ω_c such that $\Phi^*\omega_c - c\omega_0 = O(r^{-\delta})$ for some $\delta > 0$.*

The existence part is a technical improvement of known results [23, 33, 50, 52, 53]. Observe that we do not require $\varepsilon > 2$, and that we exhibit the existence of a parameter c in each Kähler class.

Uniqueness was known assuming that the two metrics in question are $O(r^{-n-\delta})$ close [23, 33, 52], but this limits applications because AC Calabi-Yau metrics often decay more slowly than $O(r^{-n})$. We pursue a completely different approach here that allows us to relax $O(r^{-n-\delta})$ to $O(r^{-\delta})$.

New idea to deal with uniqueness. One version of this idea asserts the following: *A harmonic function of strictly less than quadratic growth on a complete* AC *Kähler manifold with* $\mathrm{Ric} \geq 0$ *is of necessity pluriharmonic* (Corollary 3.9). Since all harmonic growth rates are at least 1, this implies in particular that there exists at most one such rate in $[1, 2)$. The proof makes heavy use of the AC structure (the key ingredient here is a lemma from Cheeger-Tian [16]); contrary to a claim in Li [38], the statement is in fact false for less than maximal volume growth. The same idea turns out to be useful in other settings as well, e.g. when dealing with isolated conical singularities [31].

Examples: General picture. Van Coevering [51] pointed out that AC Kähler manifolds are affine up to blowing down finitely many compact analytic sets. We then have the following rough general picture, which we will flesh out in Sections 4–5 by looking at some (extreme) special cases.

• If the complex structures on M and C converge sufficiently fast (say at least $O(r^{-2n})$), then M carries a $b^2(M)$-dimensional family of AC Ricci-flat Kähler metrics of generic decay r^{-2} towards the cone whose leading terms are harmonic $(1,1)$-forms on C. This family then contains a distinguished d-dimensional subfamily of Ricci-flat metrics with leading term $i\partial\bar{\partial}r^{2-2n}$ whose Kähler forms lie in the image of $H^2_c(M) \hookrightarrow H^2(M)$. Here d is the number of irreducible compact divisors in M.

• If the complex structures converge more slowly, then we still have a $b^2(M)$-dimensional family of metrics, but we know little about their fine asymptotics. The problem of finding the precise decay rates, e.g. in terms of the algebraic structure of M, does not seem to be a computational one.

Crepant resolutions. If M is a crepant resolution of C, then M and C are biholomorphic outside a compact set. We begin Section 4 by clarifying what is known about AC Ricci-flat metrics on such resolutions, where the general theory is now fairly complete thanks to [23, 33, 52, 53].

We then apply our existence and uniqueness theory to obtain a new class of Ricci-flat metrics on certain *small* resolutions of Calabi-Yau cones associated with partial flag manifolds. For example, if $\ell = \frac{n}{n+1-k} \in \mathbb{N}$, then we will see that the total space of $Q^* \otimes (\det T)^\ell$ carries a 1-parameter family of AC Ricci-flat metrics of rate r^{-2} and of cohomogeneity 1 under $\mathrm{SU}(n+1)$, where T denotes the rank k tautological bundle over the Grassmannian $G(k, n+1)$ and $Q = \underline{\mathbb{C}}^{n+1}/T$. This recovers Calabi's hyper-Kähler metrics on $T^*\mathbb{P}^n$ [11] if $k = 1$, but our discussion here is entirely PDE based.

Affine smoothings. At the opposite end of the spectrum, we can assume that M is already affine. Then M and C are no longer biholomorphic at infinity, and finding the precise decay rates becomes nontrivial; see Section 5. Of course, $d = b^2_c(M) = 0$ here. Moreover, $b^2(M) = 0$ if M is a complete intersection, so that we then obtain a *unique* AC Calabi-Yau metric up to scaling. In this case, we present an algorithm that at least provides us with a definite estimate on the decay rate.

If M is the standard smoothing of the ordinary double point in $\mathbb{C}^{n+1}$, we recover Stenzel's metric on T^*S^n and our estimate of the rate is $-2\frac{n}{n-1}$. In this very particular example, we are then able to show that this rate is in fact *optimal.* This contradicts Theorem 0.16 of Cheeger-Tian [16]; compare Remark 5.10 and see Remark 5.13 for details. The theoretical parts of [16] remain unaffected.

1.3. **Preliminaries.**

1.3.1. *Riemannian cones.* For us, the definition of a Riemannian cone will take the following form.

Definition 1.5. Let (L, g) be a compact connected Riemannian manifold. The *Riemannian cone C* with link L is defined to be $\mathbb{R}^+ \times L$ with metric $g_0 = dr^2 + r^2 g$ up to isometry. The radius function r is then characterized intrinsically as the distance from the apex in the metric completion.

Suppose that we are given a Riemannian cone (C, g_0) as above. Let (r, x) be polar coordinates on C, where $x \in L$, and for $t > 0$, define a map

$$\nu_t : L \times [1, 2] \ni (r, x) \mapsto (tr, x) \in L \times [t, 2t].$$

One checks that $\nu_t^*(g_0) = t^2 g_0$ and $\nu_t^* \circ \nabla_0 = \nabla_0 \circ \nu_t^*$, where ∇_0 is the Levi-Civita connection of g_0. Using these facts, one can prove the following basic lemma which will be useful in Section 5.

Lemma 1.6. *Suppose that $\alpha \in \Gamma((TC)^{\otimes p} \otimes (T^*C)^{\otimes q})$ satisfies $\nu_t^*(\alpha) = t^k \alpha$ for every $t > 0$ for some $k \in \mathbb{R}$. Then $|\nabla_0^l \alpha|_{g_0} = O(r^{k+p-q-l})$ for all $l \in \mathbb{N}_0$.*

We shall say that "$\alpha = O(r^\lambda)$ with g_0-derivatives" whenever $|\nabla_0^k \alpha|_{g_0} = O(r^{\lambda - k})$ for every $k \in \mathbb{N}_0$. We will then also say that α has "rate at most λ", or sometimes, for simplicity, "rate λ", although it should be understood that the rate is really the infimum of all λ for which this holds.

1.3.2. *Kähler and Calabi-Yau cones.* Boyer-Galicki [9] is a comprehensive reference here.

Definition 1.7. A *Kähler cone* is a Riemannian cone (C, g_0) such that g_0 is Kähler, together with a choice of g_0-parallel complex structure J_0. This will in fact often be unique up to sign. We then have a Kähler form $\omega_0(X, Y) = g_0(J_0 X, Y)$, and $\omega_0 = \frac{i}{2}\partial\bar\partial r^2$ with respect to J_0.

We call a Kähler cone "quasiregular" if the Reeb field $J\partial_r$ on its link generates an S^1-action (and, in particular, "regular" if this S^1-action is free), and "irregular" otherwise.

Theorem 1.8. *For every Kähler cone (C, g_0, J_0), the complex manifold (C, J_0) is isomorphic to the smooth part of a normal algebraic variety $V \subset \mathbb{C}^N$ with one singular point. In addition, V can be taken to be invariant under a $\mathbb{C}^*$-action $(t, z_1, ..., z_N) \mapsto (t^{w_1} z_1, ..., t^{w_N} z_N)$ such that all $w_i > 0$.*

This can be deduced from arguments written down by van Coevering in [54, §3.1].

Definition 1.9. We say that (C, g_0, J_0, Ω_0) is a *Calabi-Yau cone* if

(i) (C, g_0, J_0) is a Ricci-flat Kähler cone of complex dimension n,
(ii) the canonical bundle K_C of C with respect to J_0 is trivial, and
(iii) Ω_0 is a nowhere vanishing section of K_C with $\omega_0^n = i^{n^2} \Omega_0 \wedge \bar\Omega_0$.

We defer any discussion of Ricci-flat Kähler manifolds with torsion canonical bundle to [18].

1.3.3. *Calabi ansatz.* The construction of Calabi-Yau cones, or Sasaki-Einstein manifolds, is in itself a highly nontrivial problem. See Sparks [48] for an excellent recent survey.

The most elementary construction, originating in Calabi's paper [11], states that *regular* Calabi-Yau cones are classified by Kähler-Einstein Fano manifolds; see LeBrun [37, Proposition 3.1]. This idea will be useful for us in Section 5, but see also Example 4.1 for Calabi's original application.

The essence of the Calabi ansatz can be summarized as follows: If D is Kähler-Einstein Fano, then for every integer $k > 0$ dividing $c_1(D)$, there exists a regular Calabi-Yau cone structure on the space $(\frac{1}{k}K_D)^\times$, the blowdown of the zero section of the k-th root of K_D. If D has dimension $n-1$, then the radius function of the Ricci-flat cone metric is given by $r^n = \|\cdot\|_k^k$, where $\|\cdot\|_k$ denotes the Hermitian norm on $\frac{1}{k}K_D$ naturally induced from the Kähler-Einstein metric on D.

Definition 1.10. The *Fano index* of a Fano manifold D is the divisibility of K_D in $\mathrm{Pic}(D)$.

This is often 1 and never more than n, with equality iff $D = \mathbb{P}^{n-1}$. The quadric in $\mathbb{P}^n$ $(n > 2)$ is the only Fano of index $n-1$, and the cases of index $n-2$ and $n-3$ are classified as well [47].

1.3.4. *Asymptotically conical Riemannian manifolds.*

Definition 1.11. Let (M, g) be a complete Riemannian manifold and let (C, g_0) be a Riemannian cone. We call M *asymptotically conical* (AC) with tangent cone C if there exists a diffeomorphism $\Phi : C \setminus K \to M \setminus K'$ with K, K' compact, such that $\Phi^* g - g_0 = O(r^{-\varepsilon})$ with g_0-derivatives for some $\varepsilon > 0$. A *radius function* is a smooth function $\rho : M \to [1, \infty)$ with $\Phi^*\rho = r$ away from K'.

We implicitly only allow for one end in this definition. This is simply to fix ideas and because, by the splitting theorem, AC manifolds with $\mathrm{Ric} \geq 0$ can only ever have one end anyway.

The notions of an AC Kähler and an AC Calabi-Yau manifold are self-explanatory, except for the caveat that we require a Calabi-Yau to have trivial canonical bundle, just as in Definition 1.9.

1.3.5. *Kähler classes on open complex manifolds.* In this paper, we are really only interested in AC Kähler manifolds. These are 1-convex, hence biholomorphic to blowups of closed analytic subsets of $\mathbb{C}^N$ with at worst finitely many normal singularities; see Appendix A.

It seems to be useful to think of a Kähler class on a 1-convex manifold M simply as a class in the de Rham group $H^2(M)$ that contains positive $(1,1)$-forms. For all we know, these classes may very well be characterized by the inequalities $\int_Y \omega^k > 0$ for all k-dimensional ($k > 0$) compact analytic subsets $Y \subset M$, as is of course the case on compact Kähler manifolds [19].

Our construction of AC Calabi-Yau metrics relies on a slightly more restrictive notion of Kähler class than this; compare Definition 2.3. However, in all the examples that we discuss in this paper, this more restrictive definition turns out to in fact agree with the naive one above.

Let us finally remark that, in all the AC examples of interest here, the link L satisfies $H^1(L) = 0$. In particular, by a standard long exact sequence, (4.1), $H^2_c(M)$ injects as a subspace into $H^2(M)$. Thus, there is no ambiguity in talking about compactly supported classes in $H^2(M)$.

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2. Existence for the complex Monge-Ampère equation

2.1. **Overview.** We prove an abstract existence result for the complex Monge-Ampère equation on AC Kähler manifolds. The existence of AC Calabi-Yau metrics can be deduced as a corollary.

The statement of the analytic result is as follows. We use standard notation concerning weighted Hölder spaces; cf. Section 2.2. Parts (i) and (ii) can be assembled from the literature, as we explain in Section 2.3. Part (iii) is new, and will be proved in Section 2.4 by reduction to Part (ii).

Theorem 2.1. *Let (M, g, J) be an* AC *Kähler manifold of complex dimension $n \geq 2$ with tangent cone (C, g_0, J_0). Let ω, ω_0 denote the associated Kähler forms. Denote by $\mathcal{P}$ the set of exceptional weights of the scalar Laplacian as defined in* (2.6)*, and let ρ be a radius function on M. We wish to solve the complex Monge-Ampère equation $(\omega + i\partial\bar{\partial}u)^n = e^f\omega^n$, where $f \in C^\infty_\beta(M)$.*

(i) *If $\beta \in (-2n - \delta, -2n)$, $0 < \delta \ll 1$, then there exists a unique solution $u \in \mathbb{R}\rho^{2-2n} \oplus C^\infty_{\beta+2}(M)$.*
(ii) *If $\beta \in (-2n, -2)$, then there exists a unique solution $u \in C^\infty_{\beta+2}(M)$.*
(iii) *If $\beta \in (-2, 0)$ and $\beta + 2 \notin \mathcal{P}$, then there exists a solution $u \in C^\infty_{\beta+2}(M)$.*

Remark 2.2. (i) In the first case, the constant in front of ρ^{2-2n} can be determined by integrating the identity $(e^f - 1)\omega^n = \frac{1}{2}(\Delta u)\omega^n + \binom{n}{2}(i\partial\bar{\partial}u)^2 \wedge \omega^{n-2} + ... + (i\partial\bar{\partial}u)^n$ over M.

(ii) If $\mathrm{Ric}(g_0) \geq 0$, we can take $\delta = 1$, and $\mathcal{P} \cap (0, 2) = \mathcal{P} \cap [1, 2)$ contains at most one element; cf. Remark 2.10 below. Moreover, if we even have $\mathrm{Ric}(g) \geq 0$ on M, and $\beta \in (-2n, -2)$, then our work in Section 3 implies that $u \in C^\infty_{\beta+2}(M)$ is in fact the only solution in $C^\infty_{2-\varepsilon}(M)$ for all $0 < \varepsilon \leq |\beta|$ up to adding on globally defined pluriharmonic functions.

We now explain the consequences for the existence of Calabi-Yau metrics. This result encompasses all previous such results in the literature, and in fact more. We will prove it in Section 2.5.

Definition 2.3. Let M be an open complex manifold, $K \subset M$ a compact set, $C = \mathbb{R}^+ \times L$ a cone with cone metric g_0, and $\Phi : (1, \infty) \times L \to M \setminus K$ a diffeomorphism. A class in $H^2(M)$ is said to be a *μ-almost compactly supported Kähler class* for some given $\mu < 0$ if it can be represented by a Kähler form ω on M such that $\omega - \xi = d\eta$ on $M \setminus K$ with η a smooth real 1-form on $M \setminus K$ and ξ a smooth real $(1, 1)$-form on $M \setminus K$ such that $\Phi^*\xi = O(r^{\mu-k})$ with g_0-derivatives.

Theorem 2.4. *Let (M, J) be an open complex manifold of complex dimension $n > 2$ such that K_M is trivial. Let Ω be a nowhere vanishing holomorphic volume form on M. Let L be Sasaki-Einstein with associated Calabi-Yau cone (C, g_0, J_0, Ω_0) and radius function r. Suppose that there exist $\lambda < 0$, a compact subset $K \subset M$, and a diffeomorphism $\Phi : (1, \infty) \times L \to M \setminus K$ such that*

$$\Phi^*\Omega - \Omega_0 = O(r^\lambda) \tag{2.1}$$

with g_0-derivatives. Let $\mathfrak{k} \in H^2(M)$ be a μ-almost compactly supported Kähler class. Assume that

$$\nu := \max\{\lambda, \mu\} \notin \{-2n, -2, \nu_0 - 2\}, \tag{2.2}$$

where $\nu_0 \geq 1$ denotes the smallest growth rate of a pluriharmonic function on C. Then for all $c > 0$, there exists an AC *Calabi-Yau metric g_c on M whose associated Kähler form ω_c lies in $\mathfrak{k}$ and*

$$\Phi^*\omega_c - c\omega_0 = O(r^{\max\{-2n, \nu\}}) \tag{2.3}$$

with g_0-derivatives. Moreover, if $\nu < -2n$, then there exists $\varepsilon > 0$ such that

$$\Phi^*\omega_c - c\omega_0 = const \cdot i\partial\bar{\partial}r^{2-2n} + O(r^{-2n-1-\varepsilon}). \tag{2.4}$$

Remark 2.5. If $\omega - \xi = d\eta$ globally on M in Definition 2.3, we only need Part (i) of Corollary A.3 for the proof of Theorem 2.4, rather than the more difficult Part (ii). In particular, the proof then works for $n = 2$ as well. In our applications to quasiprojective manifolds [18], $\omega = \xi$ anyway.

Remark 2.6. The parameter c corresponds to scaling if (M, J) is Stein and $\mathfrak{k} = 0$, and to the flow of a holomorphic vector field if (M, J) is a crepant resolution of the cone (C, J_0); cf. Corollary 3.14. However, we do not know whether or not c is caused by scaling and diffeomorphism in general.

2.2. **Linear analysis.** We require a definition of weighted Hölder spaces.

Definition 2.7. Let (M,g) be AC with tangent cone (C,g_0), and let ρ be a radius function.

(i) For $\beta \in \mathbb{R}$ and k a nonnegative integer, define $C^k_\beta(M)$ to be the space of continuous functions u on M with k continuous derivatives such that

$$\|u\|_{C^k_\beta} := \sum_{j=0}^{k} \sup_M |\rho^{j-\beta}\nabla^j u| < \infty.$$

Define $C^\infty_\beta(M)$ to be the intersection of the $C^k_\beta(M)$ over all $k \in \mathbb{N}_0$.

(ii) Let $\delta(g)$ be the convexity radius of g, and write $d(x,y)$ for the distance between two points x and y in M. For T a tensor field on M and $\alpha, \gamma \in \mathbb{R}$, define

$$[T]_{C^{0,\alpha}_\gamma} := \sup_{\substack{x \neq y \in M \\ d(x,y) < \delta(g)}} \left[\min(\rho(x),\rho(y))^{-\gamma} \frac{|T(x)-T(y)|}{d(x,y)^\alpha} \right],$$

where $|T(x)-T(y)|$ is defined via parallel transport along the minimal geodesic from x to y.

(iii) For $\beta \in \mathbb{R}$, k a nonnegative integer, and $\alpha \in (0,1)$, define the weighted Hölder space $C^{k,\alpha}_\beta(M)$ to be the set of $u \in C^k_\beta(M)$ for which the norm

$$\|u\|_{C^{k,\alpha}_\beta} := \|u\|_{C^k_\beta} + [\nabla^k u]_{C^{0,\alpha}_{\beta-k-\alpha}} < \infty.$$

Remark 2.8. Whether one decides to measure the asymptotics of a function $u \in C^k_\beta(M)$ in terms of the metric g or g_0 actually makes no difference.

Let (C,g_0) be a Riemannian cone with $\dim_\mathbb{R} C = m > 2$ and with link L, and let (M^m,g) be AC with tangent cone (C,g_0). Consider the Laplacian Δ on functions derived from g. Then

$$\Delta : C^{k+2,\alpha}_{\beta+2}(M) \to C^{k,\alpha}_\beta(M). \tag{2.5}$$

The main point of defining weighted Hölder spaces is that there exists a satisfactory elliptic theory associated with the mapping (2.5), which we now summarize. Our reference is Marshall [42]; a major earlier paper by Lockhart-McOwen [40] develops the theory in weighted Sobolev spaces.

Theorem 2.9 ([42, Theorem 6.10])**.** *Define the set of exceptional weights*

$$\mathcal{P} := \left\{ -\frac{m-2}{2} \pm \sqrt{\frac{(m-2)^2}{4} + \mu} : \mu \geq 0 \text{ is an eigenvalue of } \Delta_L \right\}. \tag{2.6}$$

Then the operator (2.5) *is Fredholm if and only if* $\beta + 2 \notin \mathcal{P}$.

The elements of $\mathcal{P} \cap \mathbb{R}^+$ are precisely the possible growth rates of polynomial harmonic functions on C and on M. For instance, if $C = \mathbb{R}^m$, then $\mathcal{P} = \mathbb{N}_0 \cup (2-m-\mathbb{N}_0)$. In general, the elements of $\mathcal{P}$ are spread symmetrically about the point $\frac{2-m}{2}$ and lie outside the interval $(2-m,0)$, with $2-m$ and 0 being exceptional. The weight $2-m$ is associated with the Green's function on M.

Remark 2.10. By the Lichnerowicz-Obata theorem, if (C,g_0) has nonnegative Ricci curvature and is not isometric to $\mathbb{R}^m$, then $\mathcal{P} \cap [1-m,1] = \{2-m,0\}$. Moreover, as we noted in Remark 2.2(ii), if (C,g_0) is in addition Kähler, then there exists at most one exceptional weight in the interval $(1,2)$: the smallest possible growth rate of a pluriharmonic function; see Corollary 3.6.

The main point for us then is the following existence result:

Theorem 2.11 ([42, Theorem 6.10])**.** *If* $\beta + 2 \in (2-m,\infty) \setminus \mathcal{P}$*, then the map* (2.5) *is surjective.*

Using Theorem 2.9, this is essentially dual to the fact that (2.5) is injective for $\beta < -2$, which is in turn clear from the maximum principle. If $f \in C^{k,\alpha}_\beta(M)$ with $\beta < -m$, then $\Delta u = f$ is of course still solvable, but the leading term of u will typically be a constant multiple of r^{2-m}.

2.3. **Existence if the right-hand side decays fast.** We will now explain how Parts (i) and (ii) of Theorem 2.1 follow by using well-known methods. See [52] for a more detailed write-up.

If M is ALE, i.e. if the tangent cone is flat, both parts were proved by Joyce [33, Theorem 8.5.1]. Joyce's strategy for Part (ii) carries over to the general AC case. We will thus assume for now that $\beta \in (-2n, -2)$, and explain the extra arguments needed to deal with Part (i) later. We make use of the continuity method to solve the equations $(\omega + i\partial\bar{\partial}u_t)^n = e^{tf}\omega^n$ in $C^{k,\alpha}_{\beta+2}(M)$ for all k. Openness follows from Theorem 2.11 because $\beta \in (-2n, -2)$; closedness follows from a priori estimates.

The key to these estimates is to establish the following Sobolev inequality: For each AC manifold (M, g) of real dimension $m > 2$, there exists a constant $C < \infty$ such that

$$\left(\int_M |u|^{\frac{2m}{m-2}}\, d\mathrm{vol}\right)^{\frac{m-2}{m}} \leq C \int_M |\nabla u|^2\, d\mathrm{vol} \tag{2.7}$$

for all $u \in C^\infty_0(M)$. Once this estimate is known, Moser iteration with weights yields a $C^0_{\beta+2}$ bound on u_t. Yau's method then provides a uniform bound on $i\partial\bar{\partial}u_t$, hence on $[i\partial\bar{\partial}u_t]_{C^{0,\alpha}}$ by Evans-Krylov, which together with the $C^0_{\beta+2}$ bound from before can be bootstrapped into a $C^\infty_{\beta+2}$ bound.

In the ALE case, Joyce notes that (2.7) can be deduced from elliptic theory in weighted Sobolev spaces; details can be found in Pacini's work [45, §13], which in fact deals with the general AC case. Tian and Yau [50, §3] proved (2.7) in a number of special cases by using the solution of the Plateau problem and the Michael-Simon Sobolev inequality. General and completely elementary proofs were given by van Coevering [52, §2.2] and by the second author [32, Theorem 1.2].

This completes our discussion of Part (ii). We can now explain the proof of Part (i). If $\beta < -2n$, then Part (ii) tells us that there exists a unique solution $u = O_\varepsilon(r^{2-2n+\varepsilon})$. Thus, by expanding the Monge-Ampère equation, $\Delta_{g_0} u = O(r^\beta)$ on $M \setminus K$ if $\beta > \max\{-4n, -2n + \lambda\}$. In the ALE case, we can invoke standard Green's function estimates on $\mathbb{C}^n$ to deduce that $u = \mathit{const} \cdot r^{2-2n} + O(r^{\beta+2})$ if, in addition, $\beta > -2n - 1$. This is generalized to AC manifolds in [52], using abstract heat kernel estimates. Alternatively, one can expand the relation $\Delta_{g_0} u = O(r^\beta)$ according to eigenfunctions of the Laplacian on each slice of the cone, and then analyze the resulting radial ODE's.

2.4. **Existence if the right-hand side decays slowly.** The purpose of this section is to explain how Part (iii) of Theorem 2.1 can be reduced to Part (ii) by using the following lemma.

Lemma 2.12. *Suppose that $\beta \in (-2, 0)$ and $\beta + 2 \notin \mathcal{P}$. If $f \in C^\infty_\beta(M)$, then there exists a function $u_1 \in C^\infty_{\beta+2}(M)$ such that $\omega + i\partial\bar{\partial}u_1 > 0$ and $(\omega + i\partial\bar{\partial}u_1)^n = e^{f-f_1}\omega^n$ for some $f_1 \in C^\infty_{2\beta}(M)$.*

Let us take this for granted for now. Then $\omega_1 := \omega + i\partial\bar{\partial}u_1$ is an AC Kähler metric again. If, by chance, $2\beta < -2$, then Part (ii) of Theorem 2.1 tells us that $(\omega_1 + i\partial\bar{\partial}u_2)^n = e^{f_1}\omega_1^n$ is solvable with $u_2 \in C^\infty_{2\beta+2}(M)$, and $u := u_1 + u_2$ will be a solution to the original equation. On the other hand, if we still have $2\beta \geq -2$, then Lemma 2.12 allows us to construct a function $u_2 \in C^\infty_{2\beta+2+\varepsilon}(M)$ for all $\varepsilon > 0$ such that $(\omega_1 + i\partial\bar{\partial}u_2)^n = e^{f_1-f_2}\omega_1^n$, where $f_2 \in C^\infty_{4\beta+\varepsilon}(M)$ for all $\varepsilon > 0$. It is then clear that we can proceed iteratively until the rate of the error drops below -2.

Proof of Lemma 2.12. We identify $M \setminus K$ and $(1, \infty) \times L$ via Φ. Let $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ be smooth with $\eta(t) = 0$ for $t \leq 1$ and $\eta(t) = 1$ for $t \geq 2$. Given $R > 1$, define $\eta_R : M \to \mathbb{R}$ by $\eta_R := \eta \circ (r/R)$. Since $\Delta : C^\infty_{\beta+2}(M) \to C^\infty_\beta(M)$ is surjective by Theorem 2.11, we can find a function $\hat{u}_1 \in C^\infty_{\beta+2}(M)$ solving the equation $\Delta\hat{u}_1 = 2f$ on M (it would be enough to solve this away from a large compact set). We now claim that $u_1 := \eta_R\hat{u}_1$ has all the desired properties if $R \gg 1$.

As for positivity of the closed $(1,1)$-form $\omega + i\partial\bar{\partial}u_1$, consider that

$$i\partial\bar{\partial}u_1 = \eta_R i\partial\bar{\partial}\hat{u}_1 + i\frac{\eta'}{R}\left(\partial\hat{u}_1 \wedge \bar{\partial}r + \partial r \wedge \bar{\partial}\hat{u}_1\right) + i\hat{u}_1\left(\frac{\eta'}{R}\partial\bar{\partial}r + \frac{\eta''}{R^2}\partial r \wedge \bar{\partial}r\right).$$

The length of this form is $O(R^\beta)$ because $\hat{u}_1 \in C^\infty_{\beta+2}(M)$, $r \in C^\infty_1(M \setminus K)$, and $\eta' \circ (r/R)$ vanishes unless $r \in [R, 2R]$. Thus, $\sup|i\partial\bar{\partial}u_1| \to 0$ as $R \to \infty$, and so $\omega + i\partial\bar{\partial}u_1 > 0$ if $R \gg 1$, as desired.

As for the volume form condition, observe that, for $r > 2R$, we have

$$\begin{aligned}(\omega + i\partial\bar\partial u_1)^n &= (1 + \frac{1}{2}\Delta\hat{u}_1)\omega^n + \binom{n}{2}(i\partial\bar\partial\hat{u}_1)^2 \wedge \omega^{n-2} + \dots + (i\partial\bar\partial\hat{u}_1)^n \\ &= (1 + f + O(r^{2\beta}))\omega^n,\end{aligned}$$

so that the lemma follows with $f_1 = f - \log(1 + f + O(r^{2\beta})) = O(r^{2\beta})$ at infinity. □

Remark 2.13. Section 2 of [50] is devoted to a similar "preconditioning", in which a certain initial AC Kähler metric gets modified step by step to improve the decay of its Ricci potential. We have found this part of [50] difficult to follow. Our proof here exhibits a clean way of doing the iterative improvement if one is willing to invest some basic elliptic theory in weighted Hölder spaces.

2.5. **Application to Calabi-Yau metrics.** This section is dedicated to the proof of Theorem 2.4. We begin with two preliminary lemmas.

Lemma 2.14. *In the setting of Theorem 2.4, we have $\Phi^*J - J_0 = O(r^\lambda)$ with g_0-derivatives.*

Proof. We again identify $M \setminus K$ and $(1,\infty) \times L$ via Φ, and we tacitly allow ourselves to work away from increasingly large compact sets whenever necessary. All metric quantities and operations will be the ones associated with g_0. We only prove C^0 decay, using linear algebra as in [20, §2].

Let us begin by defining the complex vector spaces

$$U_0 := \wedge^{1,0}_{J_0}(T^{\mathbb{C}}_x M)^*, \;\; U := \wedge^{1,0}_{J}(T^{\mathbb{C}}_x M)^*, \;\; V := (T^{\mathbb{C}}_x M)^*,$$

for each $x \in M \setminus K$. Then $U_0, U \subset V$ with natural Hermitian structures induced by g_0.

Claim: There exists a $\mathbb{C}$-linear map $\mu : U_0 \to \bar{U}_0$, $\|\mu\| \leq C|\Omega - \Omega_0|$, with $U = \{u + \mu(u) : u \in U_0\}$.

Let us first see how this implies C^0 decay for J. Any $\sigma \in U_0$ can obviously be written as

$$\sigma = [u + \mu(u)] + [v + \bar\mu(v)], \;\; u := (1 - \bar\mu\mu)^{-1}(\sigma), \;\; v := -\mu(u),$$

bearing in mind the fact that $\|\mu\| \ll 1$, so that $1 - \bar\mu\mu$ is indeed invertible. As a consequence,

$$|(J - J_0)\sigma| = |i(u + \mu(u)) - i(v + \bar\mu(v)) - i\sigma| \leq C|\Omega - \Omega_0||\sigma|,$$

which implies what we need to know.

Proof of the claim. Set $W := \wedge^{n+1}V$ and define maps $T, T_0 : V \to W$ by $T_{(0)}\alpha = \Omega_{(0)} \wedge \alpha$. Then

$$\ker T_0 = U_0, \;\; \ker T = U, \;\; (\ker T_0)^\perp = \bar{U}_0.$$

Let π_0 denote the g_0-orthogonal projection from W onto $\operatorname{im} T_0$, and define $T' := \pi_0 \circ T$. Obviously $\|T - T_0\| \leq C|\Omega - \Omega_0| \ll 1$ and $\dim\ker T_0 = \dim\ker T$, so that π restricts to an isomorphism from $\operatorname{im} T$ onto $\operatorname{im} T_0$, and T' restricts to an isomorphism from $\bar{U}_0$ onto $\operatorname{im} T_0$. It is now clear that

$$\mu := -(T'|_{\bar{U}_0})^{-1} \circ (T'|_{U_0})$$

presents U as a graph over U_0. Moreover,

$$\begin{aligned}\|T'|_{U_0}\| &= \|(T' - T_0)|_{U_0}\| \leq C|\Omega - \Omega_0|, \\ \|(T'|_{\bar{U}_0})^{-1}\| &\leq \|(T_0|_{\bar{U}_0})^{-1}\| + C|\Omega - \Omega_0| \leq C,\end{aligned}$$

and so $\|\mu\|$ does indeed decay in the desired fashion. □

Given Lemma 2.14 and Theorem 2.1, we can now follow [51, §4.2]. The only difference is that we make use of the fact that $r^{2\alpha}$ is strictly plurisubharmonic for all $\alpha > 0$, not just $\alpha = 1$.

Lemma 2.15. *Let (M, g, J) be a Kähler manifold and let (C, g_0, J_0) be a Kähler cone with radius r such that there exist $\lambda < 0$, a compact $K \subset M$, and a diffeomorphism $\Phi : \{r > 1\} \to M \setminus K$, with $|\nabla^k_0(\Phi^*J - J_0)|_0 = O(r^{\lambda-k})$ for $k = 0, 1$. Then, for all $\alpha > 0$, M admits a smooth plurisubharmonic function h_α which is strictly plurisubharmonic and equal to $(r \circ \Phi^{-1})^{2\alpha}$ outside a compact K_α.*

Proof. We identify $M \setminus K$ and $\{r > 1\}$ via Φ. Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be smooth with $\psi', \psi'' \geq 0$ and

$$\psi(t) = \begin{cases} (R+2)^2 & \text{if } t < (R+1)^2, \\ t & \text{if } t > (R+3)^2, \end{cases}$$

for some $R = R_\alpha > 1$ to be specified later. Then $h_\alpha := \psi \circ r^{2\alpha} : M \to \mathbb{R}^+$ satisfies

$$i\partial\bar{\partial} h_\alpha = \begin{cases} 0 & \text{on } K \cup \{1 < r < R+1\}, \\ \psi'' i\partial r^{2\alpha} \wedge \bar{\partial} r^{2\alpha} + \psi' i\partial\bar{\partial} r^{2\alpha} & \text{on } \{r > R\}. \end{cases}$$

Since $i\partial u \wedge \bar{\partial} u \geq 0$ with respect to J for any smooth real-valued function u, it suffices to prove that $i\partial\bar{\partial} r^{2\alpha} > 0$ with respect to J on $\{r > R\}$, provided that $R \gg 1$. Indeed, on $M \setminus K$,

$$|i\partial\bar{\partial} r^{2\alpha} - i\partial_0\bar{\partial}_0 r^{2\alpha}|_0 = \frac{1}{2}|d((J - J_0)dr^{2\alpha})|_0 = O(r^{2\alpha-2+\lambda}).$$

This is of lower order compared to $i\partial_0\bar{\partial}_0 r^{2\alpha}$, which one easily checks is positive with respect to J_0. Using $J - J_0 = O(r^\lambda)$ again, this shows that $i\partial\bar{\partial} r^{2\alpha}$ is positive with respect to J for $R \gg 1$. □

Proof of Theorem 2.4. We identify $M \setminus K$ and $(1, \infty) \times L$ via Φ and allow ourselves to work off of increasingly large compact sets if convenient. By Lemma 2.14, $J - J_0 = O(r^\lambda)$ with g_0-derivatives. As a consequence, by Lemma 2.15, M admits smooth plurisubharmonic exhaustions h_α ($\alpha > 0$) that are strictly plurisubharmonic and equal to $r^{2\alpha}$ outside a compact set. Moreover, for all $k \in \mathbb{N}_0$,

$$|\nabla_0^k(i\partial\bar{\partial} r^2 - \omega_0)|_{g_0} = O(r^{\lambda-k}). \tag{2.8}$$

By assumption, we have a Kähler form ω on M and a smooth real $(1,1)$-form ξ on $M \setminus K$ with $|\nabla_0^k \xi|_0 = O(r^{\mu-k})$ such that $\omega - \xi$ is d-exact on $M \setminus K$. Thus, by Corollary A.3(ii), $\omega - \xi = -i\partial\bar{\partial} u$ on $\{r > R\}$ for some $R > 1$ and some smooth real-valued function u. Fix $\alpha \in (0,1)$. We can assume that $h_\alpha = r^{2\alpha}$ as well as $h_1 = r^2$ on $\{r > R\}$, and that both functions are strictly plurisubharmonic on this region. Also, fix a cutoff function $\zeta : M \to \mathbb{R}$ with

$$\zeta(x) = \begin{cases} 0 & \text{if } r(x) < 2R, \\ 1 & \text{if } r(x) > 3R, \end{cases}$$

and define $\zeta_S(x) := \zeta(x/S)$ in the obvious way for $S > 2$. Given $c > 0$, we now construct

$$\hat{\omega} := \omega + i\partial\bar{\partial}(\zeta u) + Ci\partial\bar{\partial}((1 - \zeta_S)h_\alpha) + ci\partial\bar{\partial} h_1,$$

with C and S to be determined. Note that $\hat{\omega} = \omega + Ci\partial\bar{\partial} h_\alpha + ci\partial\bar{\partial} h_1 \geq \omega > 0$ on $K \cup \{1 < r < 2R\}$ because h_α and h_1 are plurisubharmonic; $\hat{\omega} = \xi + Ci\partial\bar{\partial} h_\alpha + ci\partial\bar{\partial} h_1 > 0$ on $\{3R < r < 2SR\}$, after increasing R if necessary, because $|\xi| = O(r^\mu)$; $\hat{\omega} > 0$ on $\{3SR < r\}$ for the same reason; $\hat{\omega} > 0$ on $\{2R \leq r \leq 3R\}$ by compactness if R is now fixed and C is made large enough; and finally, $\hat{\omega} > 0$ on $\{2SR \leq r \leq 3SR\}$ if $S \gg 1$ depending on all previous choices because h_α is of lower order compared to h_1. In conclusion, $\hat{\omega}$ is a genuine Kähler form on M with $\hat{\omega} = \xi + ci\partial\bar{\partial} r^2$ at infinity.

Let us denote the metric corresponding to $\hat{\omega}$ by $\hat{g}$. Then, from (2.8), for all $k \in \mathbb{N}_0$,

$$|\nabla_0^k(\hat{g} - cg_0)|_0 = O(r^{\nu-k}).$$

As a result, $(M, \hat{g}, J)$ is an AC Kähler manifold of rate ν, and $\hat{g}$ has a global Ricci potential

$$\hat{f} = \log\left(\frac{i^{n^2}\Omega \wedge \bar{\Omega}}{(\hat{\omega}/c)^n}\right) \in C_\nu^\infty(M).$$

We may now appeal to Theorem 2.1 in order to solve the complex Monge-Ampère equation

$$(\hat{\omega} + i\partial\bar{\partial}\hat{u})^n = e^{\hat{f}}\hat{\omega}^n$$

for $\hat{u}$, bearing in mind that, by Remark 2.10, $\mathcal{P} \cap (0, 2)$ is either empty or equal to $\{\nu_0\}$. □

3. Uniqueness in a given Kähler class

Let (M^n, J) be an open complex manifold with a nowhere vanishing holomorphic volume form Ω. Let ω_1, ω_2 be Ricci-flat Kähler forms in the same de Rham class on M, with volume form $i^{n^2}\Omega \wedge \bar{\Omega}$, whose associated Kähler metrics g_1, g_2 are complete. In this generality, g_1, g_2 need not even have the same volume growth. However, if one assumes in addition that g_1 and g_2 are isometric at infinity, then one might perhaps expect that $g_1 = \Phi^* g_2$ for some $\Phi \in \mathrm{Aut}(M, J, \Omega)$. One difficulty here stems from the fact that (M, J) may have a huge automorphism group. For example, any map of the form $(z, w) \mapsto (z, w + f(z))$ with $f \in \mathcal{O}(\mathbb{C})$ is an automorphism of $\mathbb{C}^2$ that preserves $dz \wedge dw$.

What one *can* prove in some cases is that $\omega_1 = \omega_2$ if $\omega_1 - \omega_2$ has rapid decay. In the ALE case, this is due to Joyce [33, Theorem 8.4.4] under the assumption that $\omega_1 - \omega_2 = O(r^{-n-\varepsilon})$. This rate is good enough for applications because ALE Calabi-Yau metrics decay like r^{-2n}. Direct extensions to the AC case were proved in [23, 52], but $O(r^{-n-\varepsilon})$ now no longer suffices for many applications. We pursue an entirely different approach here that allows us to relax $O(r^{-n-\varepsilon})$ to $O(r^{-\varepsilon})$.

Theorem 3.1. *Let (M, J) be an n-dimensional complex manifold with* AC *Kähler metrics g_1, g_2. If $\eta = \omega_1 - \omega_2$ is d-exact, $\eta \in C^\infty_{-\varepsilon}(M)$ for some $\varepsilon > 0$, $\omega_1^n = \omega_2^n$, and* $\mathrm{Ric}(g_2) \geq 0$, *then $\eta = 0$.*

Remark 3.2. Pulling back the flat metric on $\mathbb{C}^n$ by a 1-parameter subgroup of $\mathrm{SL}(n, \mathbb{C})$ shows that $\eta = O(r^{-\varepsilon})$ cannot be relaxed to $\eta = O(1)$. Theorem 3.1 is also false for volume growth $2n - 1$ in place of $2n$; the Taub-NUT metric will serve as a useful counterexample throughout the proof.

Remark 3.3. Carron [13] proved a Calabi-Yau type uniqueness theorem, assuming an intermediate rate of decay, on the Hilbert scheme of N points in $\mathbb{C}^2$ with a QALE hyper-Kähler metric. It would be interesting to see whether there exists a common extension of Carron's result and ours.

The rest of this section is dedicated to proving Theorem 3.1. Some conventions first:
- A 1-form α is *harmonic* if $-\Delta\alpha = (dd^* + d^*d)\alpha = 0$.
- A tensor T on a Riemannian cone is *homogeneous* if $L_{r\partial_r} T = \mu T$ for some $\mu \in \mathbb{R}$.
- The *rate* of T is the infimum of all λ such that $|\nabla^k T| = O(r^{\lambda - k})$ as $r \to \infty$ for every k.
- $\varepsilon_0 \in (0, 1)$ will denote a small constant that only depends on the manifold in question.

Also, for context, recall the following version of the Lichnerowicz-Obata theorem: If a harmonic function on a Riemannian cone with $\mathrm{Ric} \geq 0$ has positive rate, then the rate is in fact at least one, and equality is attained precisely for the linear functions on Euclidean space.

The following lemma, due to Cheeger and Tian [16], contains the key idea. A priori there would not even be a reason to expect that α is closed; yet the conclusion is that α must be exact.

Lemma 3.4 ([16, Lemma 7.27])**.** *Let α be a homogeneous harmonic* 1*-form of rate λ on a cone with* $\mathrm{Ric} \geq 0$*. If $\lambda \in [0, 1)$, then $\alpha = du$ for a homogeneous harmonic function u of rate $\lambda + 1$.*

Remark 3.5. As for harmonic functions, there are only very few possibilities for harmonic 1-forms α on a cone C *not* to be sums of homogeneous ones, or in other words, to contain any log terms at all. Working through the relevant separation of variables based on [16, (2.14), (2.15)], one finds that the only possible sources of log terms are $\dim C = 3$ and $\alpha = r^{-(1/2)}(\log r)(\kappa\, dr + 2r\, d\kappa)$ with κ a $\frac{3}{4}$-eigenfunction on the link L, or $\dim C = 4$ and $\alpha = \frac{1}{r}(\log r)\eta$ with η the radially parallel extension of a harmonic 1-form on L. Neither of these occurs if $\mathrm{Ric} \geq 0$. See also [16, p. 546].

We can draw a simple corollary which, to our knowledge, has not been observed so far.

Corollary 3.6. *Any homogeneous harmonic function of rate in* $[1, 2)$ *on a Kähler cone with* $\mathrm{Ric} \geq 0$ *is already pluriharmonic. In particular, there can be at most one harmonic rate in* $[1, 2)$*.*

Proof. Apply Lemma 3.4 to $\alpha = d^c v$ if v is harmonic. Every Kähler cone can be viewed as the total space of a negative $\mathbb{C}^*$-bundle over a compact Kähler orbifold; the polynomial growth holomorphic functions on the cone then constitute the section ring of the dual line bundle. □

We now wish to lift Lemma 3.4 from cones to AC manifolds. This requires some preparations.

Lemma 3.7. *Let (M, g) be an AC Riemannian manifold with tangent cone (C, g_0).*

(i) *Let α be a harmonic 1-form of rate λ on M. Then $\alpha = \alpha_0 + \beta$ outside a compact set, where α_0 is a harmonic 1-form of rate λ on C and β has rate at most $\lambda - \varepsilon_0$.*

(ii) *Let u_0 be a harmonic function of rate $\lambda > 0$ on C. Then there exists a harmonic function u of rate λ on M such that $u = u_0 + v$ outside a compact set, with v of rate at most $\lambda - \varepsilon_0$.*

Proof. (i) It suffices to solve the equation $\Delta_{g_0}\beta = -\Delta_{g_0}\alpha = O(r^{\lambda-2-\varepsilon_0})$ for β of rate at most $\lambda-\varepsilon_0$, defined outside a large compact set. There are no obstructions to doing this: On the cone,

$$\Delta_{g_0} = \nabla_{\partial_r}\nabla_{\partial_r} + \frac{2n-1}{r}\nabla_{\partial_r} - \frac{1}{r^2}P(\nabla_L^*\nabla_L + \mathrm{Ric})P^{-1}, \tag{3.1}$$

where P denotes radial parallel transport from the link $L = \{r = 1\}$. Now the operator $\nabla_L^*\nabla_L + \mathrm{Ric}$ acting on sections of $T^*C|_L$ has a spectral decomposition, which we can use, together with (3.1), to split up $\Delta_{g_0}\beta = -\Delta_{g_0}\alpha$ as an infinite series of ODE's. These are easy to solve individually, with the correct behavior at infinity. Summing the solutions is not difficult either because $-\Delta_{g_0}\alpha$ is smooth, so that its Fourier coefficients on each slice decay rapidly in terms of the spectral parameter.

(ii) Extend u_0 to a function $\bar{u}_0$ on M. Then $\Delta_g\bar{u}_0 \in C^\infty_{\lambda-2-\varepsilon_0}(M)$. Since we have $\lambda-2-\varepsilon_0 > -2n$, Theorem 2.11 tells us that there exists v of rate at most $\lambda-\varepsilon_0$ such that $\Delta_g v = -\Delta_g\bar{u}_0$. □

The following is then the promised extension of Lemma 3.4 from cones to AC manifolds. Notice that the Bochner formula already tells us that M does not admit any nonzero harmonic 1-forms of negative rate; this is in fact an important ingredient for the proof of the theorem.

Theorem 3.8. *Let M be an* AC *Riemannian manifold with* $\mathrm{Ric} \geq 0$. *Let α be a harmonic 1-form of rate λ on M. If $\lambda \in [0, 1)$, then $\alpha = du$ for some harmonic function u of rate $\lambda + 1$.*

Proof. By Lemma 3.7(i), $\alpha = \alpha_0 + \beta$ outside a compact set, where α_0 is harmonic of rate λ on the cone and β has rate at most $\lambda - \varepsilon_0$. Since harmonic 1-forms on C are sums of homogeneous ones by Remark 3.5, we can assume that α_0 is in fact homogeneous. Then, by Lemma 3.4, $\alpha_0 = du_0$ for a harmonic function u_0 of rate $\lambda + 1$ on C. By Lemma 3.7(ii), there exists a harmonic function u of rate $\lambda + 1$ on M such that $u = u_0 + v$ away from a compact set, with v of rate at most $\lambda + 1 - \varepsilon_0$. Thus, $\alpha = du + \hat{\alpha}$ on M, where $\hat{\alpha}$ has rate at most $\lambda - \varepsilon_0$ and is necessarily harmonic.

We now repeat the argument with $\hat{\alpha}$ in place of α for as long as the rate of the remainder stays nonnegative. Since the rate of the remainder drops by at least some universal ε_0 at each stage, we can assume that $\alpha = du + \hat{\alpha}$ after finitely many iterations, where u is harmonic of rate $\lambda + 1$ and the harmonic 1-form $\hat{\alpha}$ has a negative rate. On the other hand, since $\mathrm{Ric} \geq 0$, the Bochner formula tells us that $\Delta|\hat{\alpha}|^2 \geq 0$ globally on M. Thus, $\hat{\alpha} = 0$ by the strong maximum principle. □

We again have a direct consequence, which we need for the proof of Theorem 3.1.

Corollary 3.9. *Any harmonic function of rate strictly less than* 2 *on an* AC *Kähler manifold with* $\mathrm{Ric} \geq 0$ *must already be pluriharmonic.*

Remark 3.10. Corollary 5 in Li [38] states that $o(r^2)$ harmonic functions on *every* complete Kähler manifold with $\mathrm{Ric} \geq 0$ are pluriharmonic. This is false: Consider the Taub-NUT manifold, which is complete hyper-Kähler of real dimension 4 with cubic volume growth, with a triholomorphic Killing field X that rotates the circles at infinity [44]. Fix a parallel complex structure J and define u by $du = X \lrcorner\, \omega$. Then u is harmonic of linear growth (asymptotic to the linear function on $\mathbb{R}^3$ determined by $J \in S^2 \subset \mathbb{R}^3$), but not J-pluriharmonic because $i\partial\bar{\partial}u = \nabla X$ is a nonzero L^2-harmonic 2-form.

The proof of Li's theorem relies on an incorrect Bochner formula [38, (14)], whose correct version [39, Lemma 4.1] involves the full curvature tensor. As a consequence, Li's theorem does hold under the assumption of nonnegative bisectional rather than Ricci curvature.

A second consequence of Theorem 3.8 which is important for us is the following $i\partial\bar{\partial}$-lemma.

Theorem 3.11. *Let M be an* AC *Kähler manifold with* $\mathrm{Ric} \geq 0$. *Let η be a d-exact real $(1,1)$-form on M with $\eta \in C^\infty_{-\varepsilon}(M)$ for some $\varepsilon \in (0, \varepsilon_0)$. Then $\eta = i\partial\bar\partial u$ with $u \in C^\infty_{2-\varepsilon}(M)$.*

Proof. (1) We first wish to prove that $\eta = d\zeta$, where $\zeta \in C^\infty_{1-\varepsilon}(M)$. By assumption, $\eta = d\nu$ for some 1-form ν on M. Let K be a large compact set, let χ be a function with $\chi \equiv 0$ on K and $\chi \equiv 1$ on the complement of some larger compact set, and put $\tilde\nu = \chi\nu$ and $\tilde\eta = d\tilde\nu$. It suffices to construct a 1-form $\tilde\zeta \in C^\infty_{1-\varepsilon}(M)$ such that $\tilde\zeta \equiv 0$ on K and $d\tilde\zeta = \tilde\eta$ because then we can set $\zeta = \nu - \tilde\nu + \tilde\zeta$.

We write the tangent cone of M as $(0,\infty) \times L$ and indicate the degree of a form by a subscript. Then $\tilde\eta = \tilde\eta_1 \wedge dr + \tilde\eta_2$, where $\tilde\eta_i$ is a 1-parameter family of i-forms on L that vanish identically for all r less than some large constant. Similarly, $\tilde\nu = \tilde\nu_0\, dr + \tilde\nu_1$, and our assumption that $d\tilde\nu = \tilde\eta$ can be rewritten as $d_L\tilde\nu_0 + \partial_r\tilde\nu_1 = \tilde\eta_1$ and $d_L\tilde\nu_1 = \tilde\eta_2$. Consequently, we need to find new solutions $\tilde\zeta_0, \tilde\zeta_1$ to these equations that still vanish unless $r \gg 1$, but with growth control at infinity.

The key is to notice that $H^1(L,\mathbb{R}) = 0$ because $\mathrm{Ric}_L > 0$. Rescaling and applying Hodge theory with parameters thus yields a unique solution $\tilde\zeta_1$ to the second equation, with $d_L^*\tilde\zeta_1 = 0$, $\tilde\zeta_1 \equiv 0$ unless $r \gg 1$, and $\tilde\zeta_1 = O(r^{1-\varepsilon})$. The first equation is now similar because there must exist $\tilde\xi_0$ with $\tilde\nu_1 = \tilde\zeta_1 + d_L\tilde\xi_0$, so that $\tilde\eta_1 - \partial_r\tilde\zeta_1 = \tilde\eta_1 - \partial_r\tilde\nu_1 + \partial_r d_L\tilde\xi_0 = d_L(\tilde\nu_0 + \partial_r\tilde\xi_0)$ is indeed d_L-exact.

(2) We have $\eta = d\zeta$ with $\zeta \in C^\infty_{1-\varepsilon}(M)$ by (1). By Theorem 2.11, the equation $\frac{1}{2}\Delta u = \bar\partial^*\zeta^{0,1}$ has a $\mathbb{C}$-valued solution $u \in C^\infty_{2-\varepsilon}(M)$. (This is unique up to $\mathbb{C}$-valued pluriharmonics by Corollary 3.9, but we don't need to make use of this fact here.) Consider the $(0,1)$-form $\xi = \zeta^{0,1} - \bar\partial u \in C^\infty_{1-\varepsilon}(M)$. This satisfies $\bar\partial\xi = \bar\partial^*\xi = 0$. Thus, in particular, $\Delta\xi = 0$. Theorem 3.8 now tells us that $\xi = dv = \bar\partial v$, where $v \in C^\infty_{2-\varepsilon}(M)$ is $\mathbb{C}$-valued, harmonic, and (obviously) antiholomorphic. This then proves that $2\mathrm{Im}(u+v)$, and in fact $2\mathrm{Im}(u)$ already, is a potential for η with the right asymptotics. □

Remark 3.12. The weighted $i\partial\bar\partial$-lemma in Theorem 3.11 can perhaps be viewed as an effective or metric version of the purely complex analytic $i\partial\bar\partial$-lemma from Corollary A.3(i).

We are now in good shape to prove the main theorem of this section.

Proof of Theorem 3.1. We have $\eta = i\partial\bar\partial u$ from Theorem 3.11, where $u \in C^\infty_{2-\varepsilon}(M)$. By assumption, $(\omega_2 + i\partial\bar\partial u)^n = \omega_2^n$. Thus, $(\Delta_{g_2}u)\omega_2^n = (i\partial\bar\partial u)^2 \wedge \psi$, where ψ, as well as its derivatives, are uniformly bounded with respect to g_2. Theorem 2.11, together with an obvious iteration argument, now tells us that $u = u' + u''$, where u' is *harmonic* on M of rate at most $2-\varepsilon$ and u'' has rate at most $2-2n$. By Corollary 3.9, u' is in fact pluriharmonic already, so that $(\omega_2 + i\partial\bar\partial u'')^n = \omega_2^n$. Since u'' goes to zero at infinity, we can now use the strong maximum principle to deduce that $u'' = 0$. □

Remark 3.13. The following shows how the above proof fails in a slightly modified situation where the statement of Theorem 3.1 is in fact false. Consider Taub-NUT with a particular choice of complex structure J as in Remark 3.10, and let u' denote the associated harmonic function of linear growth. Then the complex Monge-Ampère equation $(\omega + i\partial\bar\partial u)^2 = \omega^2$ has a nontrivial solution $u = u' + u''$ with $u'' = O(r^{-1})$ because $i\partial\bar\partial u' \neq 0$, but $\omega \wedge i\partial\bar\partial u' = 0$. The metric $\omega + i\partial\bar\partial u = \omega + O(r^{-2})$ differs from ω by flowing along the J-holomorphic vector field ∇u.

We close this section with a first application of Theorem 3.1; see also Remark 2.6.

Corollary 3.14. *Consider the* 1*-parameter family ω_c of Ricci-flat metrics in a given Kähler class $\mathfrak{k}$ constructed in Theorem 2.4.*

(i) *If $\mathfrak{k} = 0$, then $\omega_{c_2} = \frac{c_2}{c_1}\omega_{c_1}$ for all $c_1, c_2 > 0$. In this case, M is necessarily Stein.*

(ii) *At the other extreme, if M is a crepant resolution of C, then $\omega_{c_2} = \exp((\log\frac{c_2}{c_1})X)^*\omega_{c_1}$ for all $c_1, c_2 > 0$, where X denotes the lift of the holomorphic vector field $r\partial_r$ from C to M.*

The Stein property in (i) follows from the Remmert reduction theorem; see Appendix A.

Neither of the above mechanisms for creating a family of Calabi-Yau metrics in $\mathfrak{k}$ is available in general. There always exists a unique vector field X_c which is g_c-*harmonic* and asymptotic to $r\partial_r$; possibly the metrics g_c just differ by scaling and flowing along X_c viewed as a time-dependent vector field. We do not know whether or not this is the right picture in general, but see Remark 5.8.

4. Crepant resolutions

Let (C, g_0, J_0, Ω_0) be an n-dimensional Calabi-Yau cone. Recall from Theorem 1.8 that the metric completion $C \cup \{0\}$ can be naturally endowed with the structure of a normal variety V that admits affine embeddings quasihomogeneous with respect to some $\mathbb{C}^*$-action with positive weights.

In many interesting cases, the variety V will admit *crepant resolutions*. By definition, these are surjective holomorphic maps $\pi : M \to V$, where M is a smooth complex manifold with a nowhere vanishing holomorphic volume form Ω such that $\pi^*\Omega_0 = \Omega$, and π is an isomorphism onto its image away from $E = \operatorname{Exc}(\pi)$. Crepant resolutions are the most obvious class of spaces to try and apply results such as Theorem 2.4 to in order to construct complete AC Calabi-Yau metrics.

This problem has been studied extensively; we review the main points of the existing theory in Sections 4.1–4.2 and make some clarifying remarks. Section 4.3 presents some new small resolutions associated with flag varieties of compact Lie groups. Our results from Sections 2–3 allow for a quick and uniform treatment, though we suspect that an ODE-based approach must exist as well.

4.1. **Classical examples.** Every Kähler cone is naturally the total space of a negative line bundle over some compact Kähler orbifold with the zero section removed (or blown down). As such, there is always a canonical way of at least partially resolving the cone by pasting the zero section back in, though, in general, this only produces an orbifold. Based on this idea, Calabi [11] constructed many examples of AC Ricci-flat Kähler manifolds which are in fact almost explicit.

Example 4.1 (Calabi)**.** Let D be a Kähler-Einstein Fano manifold of complex dimension $n-1$, H the maximal root of the canonical bundle of D (so that $H^\iota = K_D$ with ι the Fano index of D), and $L \subset H$ the total space of the corresponding U(1)-bundle. Then $\pi_1(L) = 0$, and there exist obvious cyclic quotients $L_k = L/\mathbb{Z}_k$ for all $k \in \mathbb{N}$ such that the Kähler cones $C_k = C(L_k)$ are resolved by the total spaces M_k of the line bundles H^k. The cone C_k (equivalently, the manifold M_k) admits a global holomorphic volume form if and only if $k \mid \iota$, and this then vanishes to order $\frac{\iota}{k} - 1$ along the zero section $E_k \cong D$ in M_k. In particular, $M_k \to V_k$ is a *crepant* resolution if and only if $k = \iota$.

Calabi first lifts the Kähler-Einstein metric from D to a Sasaki-Einstein metric on L. Solving an ODE, he then constructs Ricci-flat Kähler metrics on M_k for all $k \in \mathbb{N}$ that are AC at infinity, but that have a cone angle of $2\pi\frac{\iota}{k}$ along the exceptional divisor E_k. In particular, his metric is smooth precisely when $k = \iota$. Taking D to be projective space, one obtains an ALE space with tangent cone $\mathbb{C}^n/\mathbb{Z}_n$, recovering Eguchi-Hanson for $n = 2$. Another interesting example is $D = \mathbb{P}^1 \times \mathbb{P}^1$ ($\iota = 2$), in which case C_1 is the ordinary double point, or *conifold*, $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$ in $\mathbb{C}^4$.

It is worth noting that Calabi's metrics can be universally written as $i\partial\bar\partial \sum_{k=0}^\infty c_{n,k} r^{2-2nk}$ away from a compact set; the $i\partial\bar\partial \log r$ term in [11, (4.14)] cancels with the $\Phi \circ \pi$ term in [11, (3.1)].

Let us now consider the ALE case more generally. For Γ a finite subgroup of $\mathrm{SU}(n)$ acting freely on S^{2n-1}, $\mathbb{C}^n/\Gamma$ inherits a parallel holomorphic volume form from $\mathbb{C}^n$. However, except for $\Gamma = \mathbb{Z}_n$ as above, Calabi's method would produce an AC Ricci-flat *orbifold* that resolves $\mathbb{C}^n/\Gamma$ only partially. On the other hand, there often exist more complicated resolutions that are actually smooth.

Example 4.2 (Kronheimer)**.** The singularities $\mathbb{C}^2/\Gamma$ with $\Gamma < \mathrm{SU}(2)$ admit crepant resolutions with exceptional set a Dynkin graph of rational curves. Kronheimer [36] constructed ALE hyper-Kähler metrics on these resolutions using the McKay correspondence and symplectic reduction.

If we wish to apply Theorem 2.4 to a crepant resolution $M \to V$, we can take $\lambda = -\infty$, but then the question arises as to which Kähler classes $\mathfrak{k}$ are in fact μ-almost compactly supported for some $\mu < 0$. If $\mathfrak{k} \in H^2_c(M)$, then $\mu = -\infty$ works and we obtain AC Calabi-Yau metrics with leading term $i\partial\bar\partial r^{2-2n}$. This is van Coevering's main result in [52, 53] and contains Joyce's foundational work on the ALE case [33]. Note that the classes represented by Calabi's metrics are compactly supported, and $H^2(M) = H^2_c(M)$ anyway if M is any resolution of $\mathbb{C}^n/\Gamma$. We refer to [33, 43, 53, 54] for many new examples beyond these, including crepant resolutions of irregular Calabi-Yau cones.

4.2. **Goto's theorem.** What Section 4.1 leaves open is whether an *arbitrary* class $\mathfrak{k} \in H^2(M)$ on a crepant resolution $M \to V$ is μ-almost compactly supported, assuming that $\mathfrak{k}$ contains any positive $(1,1)$-forms at all. We will now explain Goto's answer to this question [23, Theorem 5.1].

We begin with the following exact sequence, which holds on every AC Kähler manifold:

$$H^1(L) \to H^2_c(M) \to H^2(M) \to H^2(L) \to H^3_c(M). \tag{4.1}$$

Here L is the link of C, and $H^1(L) = 0$ because $\mathrm{Ric}_L > 0$. More importantly, the Bochner formula also tells us that $H^2(L) = H^{1,1}_{\mathrm{pr,b}}(L)$, the primitive basic $(1,1)$-cohomology group associated with the Sasaki structure on L; for L the canonical U(1)-bundle over a Kähler-Einstein Fano manifold D, this is nothing else but $H^{1,1}_{\mathrm{pr}}(D)$. It then follows from Poincaré duality that

$$0 \to H^2_c(M) \to H^2(M) \to H^{1,1}_{\mathrm{pr,b}}(L) \to H_{2n-3}(E), \tag{4.2}$$

where again $E = \mathrm{Exc}(\pi)$. Thus, for every closed 2-form ω on M, there exists a compact set $K \subset M$ such that $\omega = p^*\xi + d\eta$ on $M \setminus K$, where $p : C \to L$ denotes the radial projection, η is a smooth real-valued 1-form on $M \setminus K$, and ξ is some closed primitive basic $(1,1)$-form on L.

We now observe that $p^*\xi$ is a $(1,1)$-form on M because M and C are biholomorphic at infinity. Since $p^*\xi = O(r^{-2})$, this allows us to invoke Theorem 2.4 to construct AC Calabi-Yau metrics of rate $-2+\varepsilon$ for every $\varepsilon > 0$. Goto did not have the $\beta \in (-2,0)$ case of Theorem 2.1, so he needed another observation, which actually yields a better result: If ξ is a closed primitive basic $(1,1)$-form on L (ξ is then automatically harmonic), then $p^*\xi$ defines an infinitesimal Ricci-flat deformation of the cone metric ω_0, i.e. $(\omega_0 + p^*\xi)^n = e^f\omega_0^n$ with $f = O(r^{-4})$ as opposed to merely $O(r^{-2})$.

Theorem 4.3 (Goto)**.** *Let (C, g_0, J_0, Ω_0) be a Calabi-Yau cone of complex dimension $n > 2$ and let L be the link, $p : C \to L$ the radial projection, and V the normal affine variety associated to C. Let $\pi : M \to V$ be a crepant resolution and $\mathfrak{k} \in H^2(M)$ a class that contains positive $(1,1)$-forms. Then for every $c > 0$, there exists a complete Calabi-Yau metric g_c on M such that $\omega_c \in \mathfrak{k}$ and*

$$\omega_c - \pi^*(c\omega_0) = p^*\xi + O(r^{-4}), \tag{4.3}$$

where ξ is the primitive basic harmonic $(1,1)$-form on L that represents the restriction of $\mathfrak{k}$ to L. Notice that $p^*\xi = O(r^{-2})$. If $\xi = 0$, or equivalently, if $\mathfrak{k} \in H^2_c(M)$, then we even have

$$\omega_c - \pi^*(c\omega_0) = \mathit{const} \cdot i\partial\bar\partial r^{2-2n} + O(r^{-2n-1-\varepsilon}) \tag{4.4}$$

for some $\varepsilon > 0$. This is the special case covered by the earlier works [33, 52, 53].

Example 4.4 (Candelas & de la Ossa)**.** One of the very few explicitly known metrics that behave as in (4.3) with $\xi \neq 0$ lives on the so-called small resolution of the conifold [12], that is, on the total space of the rank-2 bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. This vector bundle maps to the cone C_1 over $\mathbb{P}^1 \times \mathbb{P}^1$ from Example 4.1 by contracting its zero section. In turn, we have two inequivalent birational maps from the total space of the square root of $K_{\mathbb{P}^1 \times \mathbb{P}^1}$ to the small resolution, related by a "flop".

Theorem 4.3 abstractly proves the existence of a 1-parameter family of AC Calabi-Yau metrics of rate -2 on this manifold that are all asymptotic to one fixed Calabi-Yau cone metric ω_0 on C_1 via π. This is because $H^2_c(M) = H_{2n-2}(E) = 0$ and $h^{1,1}_{\mathrm{pr}}(\mathbb{P}^1 \times \mathbb{P}^1) = 1$,[1] and because we can combine the scaling action of $\mathbb{R}^+$ on our metrics and the diffeomorphism parameter c into one single parameter that scales the exceptional $\mathbb{P}^1$, while leaving the metric unchanged at infinity.

By Theorem 3.1, this family must coincide with the explicit one from [12].

Example 4.5 (Goto)**.** Calabi's metric ω on the total space of K_D has rate $-2n$ and is $i\partial\bar\partial$-exact at infinity. Now $H^2(M) = H^2_c(M) \oplus H^{1,1}_{\mathrm{pr}}(D)$ by (4.2), and $H^2_c(M) = H_{2n-2}(E) = \mathbb{R}[\omega]$. Including one "scale" as above, ω thus moves in a $b_2(D)$-parameter family of AC Ricci-flat metrics of rate -2.

Remark 4.6. (i) By passing to a projective compactification, we will prove in [18] that the subset of $H^2(M)$ of all classes containing positive $(1,1)$-forms is an open convex cone in $H^2(M)$.

[1] This argument also shows that complex cones over rank-1 Fano manifolds have no Kähler small resolutions.

(ii) From Cheeger-Tian [16, Theorem 7.92], the $p^*\xi$ with $\xi \in \mathcal{H}^{1,1}_{\mathrm{pr,b}}(L)$ exhaust the set of leading terms of infinitesimal Ricci-flat deformations of (C, g_0) with the complex structure held fixed.

(iii) If there exist any compactly supported Kähler classes at all, then E must be a divisor. From [52, §3.4], the constant in front of $i\partial\bar\partial r^{2-2n}$ in (4.4) is positive and proportional to $\langle \mathfrak{k}^n, [M]\rangle$.

4.3. **Small resolutions associated with flag manifolds.** The theme of this section is to obtain examples of small (and therefore crepant) resolutions of Calabi-Yau cones as total spaces of vector bundles. The following lemma gives us a general tool for making constructions of this kind.

Lemma 4.7. *Let B be a compact complex manifold. Let $E \to B$ be a vector bundle of rank $r \geq 2$ and let $p : \mathbb{P}(E) \to B$ denote its projectivization. Then, for any $k \in \mathbb{N}$, the following are equivalent:*

(i) *$c_1(\mathbb{P}(E))$ is divisible by k.*

(ii) *Both r and $c_1(E) + c_1(B)$ are divisible by k.*

In this case, assume that $\mathbb{P}(E)$ is Fano and let C denote the k-th root of $K_{\mathbb{P}(E)}$ with its zero section blown down. Then C admits a small resolution by the total space of $E \otimes L$ for some $L \in \mathrm{Pic}(B)$ if and only if $k = r$. In this case, L is unique, and is in fact given by $L^{-r} = \det E \otimes \det T_B$.

Proof. The equivalence of (i) and (ii) is immediate from the identity

$$c_1(\mathbb{P}(E)) = r\xi_E + p^*(c_1(E) + c_1(B)), \tag{4.5}$$

where $\xi_E = c_1(\mathcal{O}_E(1))$. One can prove this by taking Chern characters in the exact sequences

$$0 \to T_{\mathbb{P}(E)/B} \to T_{\mathbb{P}(E)} \to p^*T_B \to 0,$$
$$0 \to \mathcal{O}_{\mathbb{P}(E)} \to \mathcal{O}_E(1) \otimes p^*E \to T_{\mathbb{P}(E)/B} \to 0,$$

the second of which is simply a family version of the usual Euler sequence for $\mathbb{P}^{r-1}$.

By construction, for any vector bundle $F \to B$, the total space of F is gotten by blowing down the fibers of $\mathbb{P}(F) \to B$ in the zero section of $\mathcal{O}_F(-1) \to \mathbb{P}(F)$. Recalling that $\mathbb{P}(E \otimes L) = \mathbb{P}(E)$ in a canonical fashion for all line bundles $L \to B$, we see that the total space of $E \otimes L$ resolves C if and only if $K_{\mathbb{P}(E)} = \mathcal{O}_{E\otimes L}(-k)$. Since $\mathbb{P}(E)$ is Fano, this is equivalent to $c_1(\mathbb{P}(E)) = k\xi_{E\otimes L}$. Recall that $c_1(E \otimes L) = c_1(E) + rc_1(L)$. Together with (4.5), this allows us to rewrite our condition as

$$(r - k)\xi_{E\otimes L} + p^*(c_1(E) + c_1(B) + rc_1(L)) = 0.$$

This is solvable if and only if $k = r$. Moreover, in this case, L as in the claim is the unique solution because p^* injects $\mathrm{Pic}(B)$ into $\mathrm{Pic}(\mathbb{P}(E))$, so that $\mathrm{Pic}(B)$ must be discrete as well. ☐

If $\mathbb{P}(E)$ is in addition Kähler-Einstein, then Theorem 4.3 immediately gives us a ρ_B-dimensional family of AC Calabi-Yau metrics of rate -2 on the small resolution $E \otimes L$. Here, ρ_B denotes the Picard rank of B, and we have used the fact that $\mathrm{Pic}(\mathbb{P}(E)) = \mathbb{Z}[\mathcal{O}_E(1)] \oplus p^*\mathrm{Pic}(B)$.

On the other hand, taking branched covers of the metrics in Example 4.5 with r-fold branching along the zero section of $K_{\mathbb{P}(E)}$, we also get a $(\rho_B + 1)$-parameter family of singular AC Calabi-Yau metrics on the total space of $\frac{1}{r}K_{\mathbb{P}(E)}$ with a cone angle of $2\pi r$ along the zero section.

Conjecture 4.8. *The singular* AC *metrics on the r-th root of $K_{\mathbb{P}(E)}$ contract along one-parameter families to the smooth* AC *metrics on $E \otimes L$ by shrinking the fibers of p in the zero section.*

We now discuss a new class of examples where a compact Lie group G acts with cohomogeneity one on the cone by holomorphic isometries, so that Theorem 3.1 immediately tells us that the AC Calabi-Yau metrics on $E \otimes L$ have cohomogeneity one under G as well. It would then be interesting to see whether the ODE that of necessity governs these metrics is explicitly solvable.

Observation 4.9. *We can construct* AC *Calabi-Yau metrics of rate -2 and cohomogeneity one on a small resolution proceeding from any compact, simply connected, semisimple Lie group G, together with a pair of parabolic subgroups $P_1 \subsetneq P_2$ such that $P_2/P_1 = \mathbb{P}^{k-1}$, where k divides $c_1(G/P_1)$.*

The main point here is that all flag manifolds G/P are Fano with a G-invariant Kähler-Einstein metric [5, §8], and $H^{\mathrm{odd}}(G/P, \mathbb{Z}) = 0$ [7], so that the bundle $G/P_1 \to G/P_2$ can be written as $\mathbb{P}(E)$ for some vector bundle $E \to G/P_2$ by [28, p. 515]. The condition on c_1 is combinatorially checkable, although in concrete examples E may already be given to us as a "tautological" vector bundle, in which case condition (ii) of Lemma 4.7 may be more practical to check.

Example 4.10. The small resolution of the conifold from Example 4.4 is recovered by taking G to be $\mathrm{SU}(2) \times \mathrm{SU}(2)$. This group, being of rank 2, has two flag manifolds: $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^1$.

Example 4.11. SU(3) also has two flag manifolds: the maximal one, $\mathrm{SU}(3)/T^2 = \mathbb{P}(T^*\mathbb{P}^2)$ (of Fano index 2, as for all groups G), and $\mathbb{P}^2$. The resulting 1-parameter family of AC Calabi-Yau metrics on $T^*\mathbb{P}^2$ is originally due to Calabi [11]; these metrics are in fact hyper-Kähler. Conjecture 4.8 was motivated by the fact that such contractions were recently constructed in this case [4, 46].

The example of SU(3) can be generalized to higher dimensions in two different ways. We refer to [5, 8.111] for computations involving $\mathrm{SU}(n+1)$ flags. The basic tool here is to label the space of all flags of the form $0 \subset V^{n_1} \subset V^{n_1+n_2} \subset ... \subset \mathbb{C}^{n+1}$ by the ordered partition $(n_1, n_2, ...)$ of $n+1$.

Example 4.12. The maximal flag variety $(1, ..., 1)$ of $\mathrm{SU}(n+1)$ fibers over every flag variety of the form $(1, ..., 1, 2, 1, ..., 1)$ with $\mathbb{P}^1$ fibers. Our construction applies here because $c_1(G/T)$ is the sum of the positive roots for every group G, which is famously divisible by 2 in the weight lattice.

Example 4.13. The Grassmannian $G(k, n+1)$ of k-planes in $\mathbb{C}^{n+1}$ is given by $(k, n+1-k)$ and has at most four projective bundles sitting over it that are also flag manifolds: $(1, k-1, n+1-k)$, $(k-1, 1, n+1-k)$, $(k, 1, n-k)$, $(k, n-k, 1)$, with fibers $\mathbb{P}^{k-1}$, $\mathbb{P}^{k-1}$, $\mathbb{P}^{n-k}$, $\mathbb{P}^{n-k}$ respectively.

It is an instructive exercise to compute the divisibility of the first Chern classes of the projective bundles using roots. The positive roots are $\lambda_i - \lambda_j$ for $i < j$ with $\sum_{i=1}^{n+1} \lambda_i = 0$. To get $c_1(G/P)$, we need to sum those roots that do not vanish on the face of the Weyl chamber associated to G/P. If we think of the ordered partition in terms of "grouping boxes",

$$n+1 = (\underbrace{\square\square\square\square\square\square}_{n_1})(\underbrace{\square\square\square\square}_{n_2})(\underbrace{\square\square\square\square\square}_{n_3})(...),$$

this just means that we need to sum all $\lambda_i - \lambda_j$ such that the i-th and j-th boxes belong to different groups. For example, we can now compute that $c_1(1, k-1, n+1-k) = (n+k)\lambda_1 + n(\lambda_2 + ... + \lambda_k)$, which is divisible by k in the weight lattice if and only if $k|n$.

However, we can give a simpler and more complete discussion via Lemma 4.7(ii). The projective bundles in question are clearly of the form $\mathbb{P}(E)$ for $E = T$, T^*, Q, Q^*, where T denotes the rank k tautological bundle and $Q = \underline{\mathbb{C}}^{n+1}/T$. Now, $\mathrm{Pic}(G(k, n+1)) = \mathbb{Z}[\det T]$ and $K_{G(k,n+1)} = (\det T)^{n+1}$, so Lemma 4.7 immediately tells us that T works iff $k|n$ (as we already know), T^* works iff $k|n+2$, Q works iff $n+1-k|n+2$, and Q^* works iff $n+1-k|n$. Moreover, we learn that the line bundle L that we need to twist E by is $\det T$ to the power $\frac{n}{k}, \frac{n+2}{k}, \frac{n+2}{n+1-k}, \frac{n}{n+1-k}$, respectively. Finally, it is then also clear that we only get a 1-parameter (scaling) family of Ricci-flat metrics.

As a special case, we can consider $G(1, n+1) = \mathbb{P}^n$. This gives us Ricci-flat metrics on the total space of $Q^* \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$ as well as on the total space of $Q \otimes \mathcal{O}_{\mathbb{P}^2}(-2)$. Now, the fiber of $Q^* \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$ over $\ell \in \mathbb{P}^n$ is naturally identified with $\mathrm{Hom}(\mathbb{C}^{n+1}/\ell, \mathbb{C}) \otimes \ell = \mathrm{Hom}(\ell, \mathbb{C}^{n+1}/\ell)^*$, the cotangent space to $\mathbb{P}^n$ at ℓ. If $n = 2$, this is in turn the same as the fiber of $Q \otimes \mathcal{O}_{\mathbb{P}^2}(-2)$, but not quite canonically so because one needs to choose an element of $\wedge^{3,0}\mathbb{C}^3$. In any case, the result is that we recover Calabi's hyper-Kähler metrics on $T^*\mathbb{P}^n$ for every $n \geq 2$. Conjecture 4.8 is in fact known to hold true in this case as well [41], generalizing the work for $n = 2$ mentioned in Example 4.11 above.

Remark 4.14. Flag manifolds and their characteristic classes are a classical topic in topology. For example, Borel-Hirzebruch [8, p. 340] noticed that the Chern numbers c_1^5 of the SU(4) flag manifolds $\mathbb{P}(T^*\mathbb{P}^3)$ and $\mathbb{P}(T\mathbb{P}^3)$ are different. Thus, the complex structures of $\mathbb{P}(T^*\mathbb{P}^3)$ and $\mathbb{P}(T\mathbb{P}^3)$ cannot be homotopic, even though their underlying smooth manifolds are trivially diffeomorphic.

5. Affine smoothings

5.1. **Overview.** We now study certain examples of AC Calabi-Yau manifolds which are, in some sense, the extreme opposites of crepant resolutions. The idea here is to realize a Calabi-Yau cone as an affine variety and to "smooth out" its singularity by adding on terms of lower degree to some of its defining equations. The cone C and its associated "smoothing" M are then diffeomorphic away from compact subsets, but they are no longer canonically diffeomorphic or even biholomorphic.

We will only be considering *regular* Calabi-Yau cones. Thus, the underlying variety $C = (\frac{1}{k}K_D)^\times$ for some $(n-1)$-dimensional Fano manifold D and some $k|c_1(D)$, and the Ricci-flat cone metric on C is lifted from a Kähler-Einstein metric on D via the Calabi ansatz; compare Section 1.3.3. By the arguments in [54] used to prove Theorem 1.8, we can assume that C is realized as an algebraic variety in $\mathbb{C}^N$ for some N, in such a way that the canonical $\mathbb{C}^*$-action on C (whose space of orbits is D) is the restriction of a diagonal $\mathbb{C}^*$-action on $\mathbb{C}^N$ with positive weights $w_1, ..., w_N > 0$.

It will be enough here to work with an intuitive notion of smoothing. The most basic example we have in mind is the 1-parameter deformation $z_1^2 + ... + z_N^2 = t$ of the ordinary double point in $\mathbb{C}^N$, with $N = n+1$. Here, as in all of the other examples we shall be considering, $w_1 = ... = w_N = 1$ because D is already projectively embedded by $\frac{1}{k}K_D$. In general, in a smoothing, the terms that are added on to the equations of C need to be of sufficiently low degree relative to $w_1, ..., w_N$.

If a cone is rigid as a variety, then no such smoothings exist. For example, $K_D^\times$ is rigid if $D = \mathbb{P}^2$ or $\mathrm{Bl}_1\mathbb{P}^2$ or any toric Fano manifold of dimension $n - 1 > 2$; compare [1, (6.3), (9.1)].

Proposition 5.1. (i) *An n-dimensional smooth affine variety M with trivial canonical bundle is a smoothing of $C = (\frac{1}{k}K_D)^\times$ as above if and only if $M = X \setminus D$ for some n-dimensional Fano manifold X of index at least* 2 *containing D as an anticanonical divisor such that $-K_X = (k+1)[D]$.*

(ii) *$H^2_c(M) = H^2(M)$ if $n = 2$, and $H^2_c(M) = 0$ if $n > 2$. In both cases, $b^2(M) = b^2(X) - 1$, and the image of the Kähler cone of X under restriction to M is the whole space $H^2(M)$.*

If $n = 2$, then $X = \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. All Fano 3-folds X of index equal to 2 are listed in [47, §12.1]. Up to deformation, every del Pezzo surface D of degree at most 7 occurs as an anticanonical divisor in at least one of these. Since the cases $D = \mathbb{P}^2$, $\mathrm{Bl}_1\mathbb{P}^2$ do not occur, we recover the fact mentioned earlier that $K_D^\times$ is not smoothable for such D. On the other hand, $D = \mathrm{Bl}_3\mathbb{P}^2$ occurs twice (in a hyperplane section of $\mathrm{Segre}(\mathbb{P}^2 \times \mathbb{P}^2)$, and in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$), and the versal deformation space of the singularity $K_D^\times$ is known to consist of two irreducible components [1, (8.2), (9.1)].

Proof of Proposition 5.1. (i) Given M, we can construct X by passing to the completion of M in the total space of $\mathcal{O}(1)$ over the weighted projective space $\mathbb{P}^{N-1}_{w_1,...,w_N}$, which naturally contains $\mathbb{C}^N \setminus 0$ as a dense open subset. Then $N_{D/X}$ is positive and $M = X \setminus D$ contains no compact analytic sets, so that D must be ample by the Nakai-Moishezon criterion (see [22, Proposition 10] for some more details on this type of argument). The relation $-K_X = (k+1)[D]$ is clear by adjunction.

For the converse, we need to show that $X \setminus D$ really is a smoothing of $C = (N^*_{D/X})^\times$. This follows from a more general construction. Given a compact complex manifold X and a smooth divisor D, let $p : L \to X$ denote the line bundle associated to D and s a defining section for D. Construct a hypersurface $X_t = \{v \in L : tv = s(p(v))\}$ for each $t \in \mathbb{C}$. This is isomorphic to X for $t \neq 0$ and to the total space of $L|_D$ for $t = 0$. Notice that any two hypersurfaces in this family intersect precisely along D in the zero section. If L is positive, we can first compactify the total space of L by adding a single point at infinity, and then remove the zero section, to construct an affine variety $V = (L^*)^\times$. Our family X_t parametrized by $t \in \mathbb{C}$ then induces a subfamily of the product family $\mathbb{C} \times V$ such that all fibers over $\mathbb{C}^*$ are copies of M whereas the central fiber is isomorphic to $(L^*|_D)^\times = C$.

(ii) We have $H^2_c(M) = H^{2n-2}(M)$ by Poincaré duality, and for $n > 2$, this vanishes because M is Stein; see [2, Theorem 1]. It follows from a version of the Gysin sequence (the long exact sequence of the pair $(X, X \setminus D)$, combined with the Thom isomorphism for $N_{D/X}$) that

$$H^0(D) \to H^2(X) \to H^2(X \setminus D) \to H^1(D).$$

Thus, $H^2(X) \to H^2(M)$ is onto with a 1-dimensional kernel. In particular, the image of the Kähler cone of X in $H^2(M)$ is an open convex cone; but this cone also contains the origin because $[D]$ is a positive line bundle and the curvature forms of Hermitian metrics on $[D]$ are exact off D. □

We now focus on smoothings of *complete intersection* Calabi-Yau cones, where, by the Lefschetz hyperplane theorem [55, §13.2.3], $b^2(M) = 0$ if $n > 2$. Our examples will in fact be diffeomorphic to the Milnor fiber of C, hence homotopy equivalent to a bouquet of n-spheres [30, Satz 1.7(iv)].

In Sections 5.2–5.4, we explain, through a few examples, how one can construct a diffeomorphism Φ from C to M away from compact sets and estimate the rate of convergence λ of the holomorphic volume forms on M and C with respect to Φ. Theorem 2.4 then yields a 1-parameter scaling family of $i\partial\bar{\partial}$-exact AC Calabi-Yau metrics of rate $\max\{-2n, \lambda\}$ on M, which, according to Theorem 3.1, contains all Calabi-Yau metrics on M that are AC with respect to Φ.

In the remaining sections, we then discuss an obvious gauge fixing issue: The rates of convergence to the cone that we compute depend upon the *choice* of a diffeomorphism Φ between M and C, and may therefore not be optimal. Section 5.5 collects some general thoughts and examples regarding this matter. In Section 5.6, we then prove that the rate $-2\frac{n}{n-1}$ that we obtain for the usual smoothing of the ordinary double point in $\mathbb{C}^{n+1}$ in Section 5.4 is in fact optimal in a precise sense.

5.2. **Example 1: Smoothings of a cubic cone in $\mathbb{C}^4$.** We first consider the cubic cone

$$C = \left\{ z \in \mathbb{C}^4 : \sum_{i=1}^{4} z_i^3 = 0 \right\}.$$

Since this variety is simply the blowdown of the zero section of $K_{\mathrm{Bl}_6\mathbb{P}^2}$, using the Calabi ansatz, we can endow it with a Calabi-Yau cone structure (g_0, Ω_0). By construction, the Kähler potential of g_0 is $\|\cdot\|^{2/3}$, where $\|\cdot\|$ is the norm induced on $K_{\mathrm{Bl}_6\mathbb{P}^2}$ by the Kähler-Einstein metric on $\mathrm{Bl}_6\mathbb{P}^2$. As for Ω_0, it is the restriction to C of the $(3,0)$-form Ω_0 on $\mathbb{C}^4$ defined implicitly by the equation

$$dz_1 \wedge \ldots \wedge dz_4 = \Omega_0 \wedge d\left(\sum_{i=1}^{4} z_i^3\right). \tag{5.1}$$

We will consider one particular smoothing of C, namely the smooth affine variety M defined by

$$M = \left\{ z \in \mathbb{C}^4 : \sum_{i=1}^{4} z_i^3 = 1 \right\}.$$

Then M carries a holomorphic volume form Ω, defined by (5.1) with Ω in place of Ω_0. Estimating the rate of convergence of Ω and Ω_0 with respect to g_0 and a diffeomorphism Φ we are yet to construct, and invoking Theorems 2.4 and 3.1, allows us to prove the following proposition.

Proposition 5.2. *For each $c > 0$, M admits an $i\partial\bar{\partial}$-exact* AC *Calabi-Yau metric ω_c such that*

$$\Phi^*\omega_c - c\omega_0 = const \cdot i\partial\bar{\partial}r^{-4} + O(r^{-7-\varepsilon}) \tag{5.2}$$

for some $\varepsilon > 0$, with $\omega_{c_2} = \frac{c_2}{c_1}\omega_{c_1}$ for all $c_1, c_2 > 0$. Moreover, these are the only Calabi-Yau metrics on M that are AC *with respect to the chosen diffeomorphism Φ.*

Remark 5.3. Even modulo biholomorphism, C admits many more affine smoothings than just M: Every smoothing of C in our sense is in particular a deformation of C. According to [34],

$$tz_1z_2z_3z_4 + \sum_{i=1}^{4} z_i^3 + \sum_{1\leq i<j<k\leq 4} t_{ijk}z_iz_jz_k + \sum_{1\leq i<j\leq 4} t_{ij}z_iz_j + \sum_{i=1}^{4} t_iz_i = \varepsilon$$

cuts out a versal deformation of C in $\mathbb{C}^{16} \times \mathbb{C}^4$. We are only interested in the smooth fibers of the subfamily given by $t = 0$, $t_{ijk} = 0$, which are homotopy equivalent to $\vee^{16}S^3$ [27, Korollar 3.10] and pairwise diffeomorphic. Proposition 5.2 holds for each of these except for the statement of the rates in (5.2). As it turns out, Ω converges to Ω_0 at rate -6 if all $t_{ij} = 0$, and at rate -3 in general.

5.2.1. *Construction of a diffeomorphism.* The idea here is to project C orthogonally onto M in $\mathbb{C}^4$ to construct a diffeomorphism Φ from the complement of a compact subset of C onto the complement of a compact subset of M. Using Lemma 5.4 below, this mapping takes the form

$$\Phi(z_1, ..., z_4) = (z_1 + \alpha(z)\bar{z}_1^2, ..., z_4 + \alpha(z)\bar{z}_4^2)$$

away from some sufficiently large closed ball $\bar{B}_R \subset \mathbb{C}^4$ of radius R centered at the origin.

Lemma 5.4. *There exists $R > 0$ and a smooth function $\alpha : \mathbb{C}^4 \setminus \bar{B}_R \to \mathbb{C}$ with $\alpha(z) \sim |z|^{-4}$ such that $\sum_{i=1}^4 (z_i + \alpha(z)\bar{z}_i^2)^3 = 1$ for every $z \in C \setminus \bar{B}_R$.*

Proof. Taking complex coordinates $z = (z_1, ..., z_4)$ on $S^7 \subset \mathbb{C}^4$, define a function f by

$$f : S^7 \times [0, \infty) \times \mathbb{C} \to \mathbb{C},\;\; f(z, r, y) = 3y \left(\sum_{i=1}^4 |z_i|^4\right) + 3y^2 \left(\sum_{i=1}^4 |z_i|^2 \bar{z}_i^3\right) + y^3 \left(\sum_{i=1}^4 \bar{z}_i^6\right) - r^3,$$

and fix any point $p = (p_1, ..., p_4) \in S^7$. Since

$$f(p, 0, 0) = 0,\;\; \frac{\partial f}{\partial y}(p, 0, 0) = 3 \sum_{i=1}^4 |p_i|^4 \neq 0,$$

the implicit function theorem asserts the existence of a unique smooth function s_p, defined in some open neighborhood $U_p \times [0, \varepsilon_p)$ of $(p, 0)$, such that $s_p(p, 0) = 0$ and $f(z, r, s_p(z, r)) = 0$. An obvious covering argument on S^7 then yields a smooth function $s : S^7 \times [0, \varepsilon) \to \mathbb{C}$ satisfying

$$s(z, 0) = 0, \quad f(z, r, s(z, r)) = 0.$$

We now set $R = \varepsilon^{-1}$ and define $\alpha : \mathbb{C}^4 \setminus \bar{B}_R \to \mathbb{C}$ by

$$\alpha(z) = \frac{1}{|z|} s\left(\frac{z}{|z|}, \frac{1}{|z|}\right).$$

The fact that α satisfies $\sum_{i=1}^4 (z_i + \alpha(z)\bar{z}_i^2)^3 = 1$ for each $z \in C \setminus \bar{B}_R$ is straightforward to verify. In order to complete the proof of the lemma, we need only show that $\alpha(z) \sim |z|^{-4}$.

To see this, observe that $\alpha(z)|z|^4 P(z) = 1$ on $\mathbb{C}^4 \setminus \bar{B}_R$, where

$$P(z) = 3\left(\sum_{i=1}^4 \frac{|z_i|^4}{|z|^4}\right) + 3(\alpha(z)|z|)\left(\sum_{i=1}^4 \frac{|z_i|^2 \bar{z}_i^3}{|z|^5}\right) + (\alpha(z)|z|)^2 \left(\sum_{i=1}^4 \frac{\bar{z}_i^6}{|z|^6}\right).$$

It then suffices to note that $\alpha(z)|z| = s(\frac{z}{|z|}, \frac{1}{|z|}) \to 0$ uniformly as $|z| \to \infty$. □

5.2.2. *Some preliminary estimates.* We next take conical coordinates (r, x) on $C = \mathbb{R}^+ \times L$, where $r(z) = \|z\|^{1/3}$ is the g_0-distance from z to the apex of C, and define diffeomorphisms

$$\nu_t : L \times [1, 2] \ni (x, r) \mapsto (x, tr) \in L \times [t, 2t].$$

We wish to write ν_t in terms of Cartesian coordinates on $\mathbb{C}^4$.

The basic observation is that $\nu_t(z) = t^\mu z$ for some $\mu > 0$ because C is homogeneously embedded, or in other words because $-K_{\mathrm{Bl}_6\mathbb{P}^2}$ is very ample. But then we must have $\mu = 3$ because

$$t\|z\|^{\frac{1}{3}} = tr(z) = r(\nu_t(z)) = \|\nu_t(z)\|^{\frac{1}{3}} = \|t^\mu z\|^{\frac{1}{3}} = t^{\frac{\mu}{3}}\|z\|^{\frac{1}{3}}.$$

It now follows immediately that $|z_i| \sim r^3$ for all i, though the precise proportionality depends upon the Kähler-Einstein metric on $\mathrm{Bl}_6\mathbb{P}^2$. In addition, $z_i = O(r^3)$ with g_0-derivatives by Lemma 1.6.

Observe that $\alpha \sim r^{-12}$. We also require estimates on $d\alpha$ and on its covariant derivatives. Since $\alpha(z)|z|^4 P(z) = 1$, we find that $|d\alpha|_{g_0} \leq Cr^{-12}(r^{-1} + |dP|_{g_0})$. Now, using the definition of P,

$$dP = Q_{-1} + (P_{-12}Q_2 + P_3 d\alpha) + (P_{-24}Q_5 + P_{-6}d\alpha),$$

where P_k (respectively Q_k) denotes a function (respectively 1-form) that we already know is $O(r^k)$. Hence, $|d\alpha|_{g_0} = O(r^{-13})$. More generally, we have that $\alpha = O(r^{-12})$ with g_0-derivatives.

5.2.3. *Computation of the asymptotics of the holomorphic volume forms.* Using these preliminary estimates, we are now able to compute the rate of convergence of the holomorphic volume forms Ω on M and Ω_0 on C with respect to our diffeomorphism Φ. First observe that we may write

$$\Omega_0 = (-1)^{i-4} \frac{dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_4}{3z_i^2}\bigg|_C,$$

and likewise for Ω, on the open subsets $C \cap \{z_i \neq 0\}$ and $M \cap \{z_i \neq 0\}$ respectively.

Next, consider the set $A_4 = \{z \in C : |z_4| > \frac{1}{3}|z| > R\}$, where $R \gg 1$ is chosen sufficiently large so that $\frac{1}{3}|z| > |\alpha(z)||z|^2$ for all $z \in C$ with $|z| > R$. Observe that $z \in A_4$ implies that $z_4 + \alpha(z)\bar{z}_4^2 \neq 0$. As a consequence, we find that in a neighborhood of any point $z \in A_4$,

$$\Phi^*\Omega = \frac{d(z_1 + \alpha(z)\bar{z}_1^2) \wedge d(z_2 + \alpha(z)\bar{z}_2^2) \wedge d(z_3 + \alpha(z)\bar{z}_3^2)}{3(z_4 + \alpha(z)\bar{z}_4^2)^2}\bigg|_C,$$

so that, using our preliminary estimates,

$$\Phi^*\Omega = \frac{(dz_1 + O(r^{-7})) \wedge (dz_2 + O(r^{-7})) \wedge (dz_3 + O(r^{-7}))}{3z_4^2(1 + O(r^{-9}))^2}\bigg|_C = \Omega_0 + O(r^{-9}).$$

This estimate works equally well on $A_i = \{z \in C : |z_i| > \frac{1}{3}|z| > R\}$ for all $i \neq 4$. In light of the fact that the A_i cover $C \setminus \bar{B}_{3R}$, we deduce that $\Phi^*\Omega - \Omega_0 = O(r^{-9})$ on $C \setminus \bar{B}_{3R}$ with g_0-derivatives.

5.3. **Example 2: Smoothings of the intersection of two quadric cones in $\mathbb{C}^5$.** Our second example is based on the complete intersection cone

$$C = \left\{ z \in \mathbb{C}^5 : f_1(z) = \sum_{i=1}^{5} z_i^2 = 0,\ f_2(z) = \sum_{i=1}^{5} \lambda_i z_i^2 = 0 \right\},$$

where $\lambda_i \neq \lambda_j$ for $i \neq j$. Let Q_1, Q_2 be the projectivizations in $\mathbb{P}^4$ of the quadrics defining C. Then $Q_1 \cap Q_2 = \mathrm{Bl}_5\mathbb{P}^2$ is an anticanonically embedded del Pezzo surface, so that C realizes the blowdown of the zero section of $K_{Q_1 \cap Q_2}$. The Calabi ansatz provides us with a Calabi-Yau cone metric g_0 on C with Kähler potential $\|\cdot\|^{2/3}$, where $\|\cdot\|$ is the norm induced on $K_{Q_1 \cap Q_2}$ by the Kähler-Einstein metric on $Q_1 \cap Q_2$. The cone C also admits a holomorphic volume form Ω_0, defined by

$$dz_1 \wedge \ldots \wedge dz_5 = \Omega_0 \wedge df_1 \wedge df_2$$

(that is, by the Poincaré residue formula), whose length with respect to g_0 is constant.

We take the smoothing $M = \{z \in \mathbb{C}^5 : f_1(z) = f_2(z) = 1\}$ of C, which also admits a holomorphic volume form Ω satisfying $dz_1 \wedge \ldots \wedge dz_5 = \Omega \wedge df_1 \wedge df_2$. Using arguments as in Section 5.2, we then construct an orthogonal projection diffeomorphism Φ of the form $\Phi(z)_i = z_i + (\alpha(z) + \bar{\lambda}_i\beta(z))\bar{z}_i$. It turns out that this yields an estimate of -6 for the rate of convergence of Ω towards Ω_0.

Proposition 5.5. *For each $c > 0$, M admits an $i\partial\bar{\partial}$-exact* AC *Calabi-Yau metric ω_c such that*

$$\Phi^*\omega_c - c\omega_0 = O_\delta(r^{-6+\delta}) \tag{5.3}$$

with g_0-derivatives for every $\delta > 0$, and such that $\omega_{c_2} = \frac{c_2}{c_1}\omega_{c_1}$ for all $c_1, c_2 > 0$. These are again the only Calabi-Yau metrics on M that are AC *with respect to the chosen diffeomorphism Φ.*

Remark 5.6. In analogy with Remark 5.3, one can show that

$$\alpha z_4^2 + \beta z_5^2 + \sum_{i=1}^{5} z_i^2 + \sum_{i=1}^{5} t_i z_i = \varepsilon_1,\ \sum_{i=1}^{5} \lambda_i z_i^2 = \varepsilon_2,$$

cuts out a versal deformation of C in $\mathbb{C}^9 \times \mathbb{C}^5$. The smoothings in our sense are exactly the smooth fibers of the subfamily given by $\alpha = \beta = 0$. They are again all diffeomorphic to each other, and are homotopy equivalent to $\vee^9 S^3$. Proposition 5.5 holds true verbatim for each such smoothing, except that the rate $-6 + \delta$ in (5.3) must be replaced by -3 if at least one of the $t_i \neq 0$.

5.4. **Example 3: The smoothing of the ordinary double point in $\mathbb{C}^{n+1}$.** Our final example is easier, and the metrics we construct are in fact not new. The cone will simply be given by

$$C = \left\{ z \in \mathbb{C}^{n+1} : f(z) = \sum_{i=1}^{n+1} z_i^2 = 0 \right\},$$

the ordinary double point in $\mathbb{C}^{n+1}$. This can be viewed as a regular Calabi-Yau cone $(\frac{1}{k}K_D)^\times$ over the hyperquadric $D \subset \mathbb{P}^n$ with its unique $\mathrm{SO}(n+1)$-invariant Kähler-Einstein metric. Notice that D has index $n-1$ if $n>2$ and is then embedded by the maximal root of $-K_D$, so that $k = n-1$. We will be working with the particular smoothing $M = \{z \in \mathbb{C}^{n+1} : f(z) = 1\} \cong T^*S^n$.

This configuration is quite special compared to the examples in Sections 5.2–5.3 for two reasons. First, the deformation space of C is only 1-dimensional now, with all smooth fibers biholomorphic to M. Bearing in mind the fact that $b^2(M) = 0$ if $n > 2$, this means that the AC Calabi-Yau metric we will construct is *rigid* in the strongest possible sense. Second, the canonical $\mathrm{SO}(n+1)$-action on $\mathbb{C}^{n+1}$ preserves both M and C, as well as the cone metric on C. So by Theorem 3.1, our AC metric on M has to be $\mathrm{SO}(n+1)$-*invariant*. The relevant normal projection map Φ from C to M that we use to estimate the rates is given by the $\mathrm{SO}(n+1)$-equivariant function $\Phi(z) = z + \frac{\bar{z}}{2|z|^2}$.

Proposition 5.7. *If $n > 2$, then M admits $i\partial\bar\partial$-exact* AC *Calabi-Yau metrics ω_c such that*

$$\Phi^*\omega_c - c\omega_0 = O(r^{-2\frac{n}{n-1}}) \tag{5.4}$$

and $\omega_{c_2} = \frac{c_2}{c_1}\omega_{c_1}$ for all $c_1, c_2 > 0$. The metrics ω_c are invariant under the obvious $\mathrm{SO}(n+1)$-*action on $\mathbb{C}^{n+1}$, and are the only Calabi-Yau metrics on M that are* AC *with respect to Φ.*

According to [49, Lemma 5], this means that we have reproduced Stenzel's metric on T^*S^n. We will revisit this point in Section 5.6 and use Stenzel's explicit formula to prove that the rate in (5.4) cannot be improved upon by deforming our chosen diffeomorphism Φ.

Remark 5.8. The case $n = 2$ is somewhat exceptional. We have $H^2_c(M) = H^2(M) = \mathbb{R}$, and every class is Kähler. Our theory then formally implies that each class in $H^2(M)$ contains a 1-parameter family of AC Calabi-Yau metrics of rate $-4+\delta$ for every $\delta > 0$. These are of course all isometric to some rescaling of Eguchi-Hanson. In particular, the rate is really -4. However, our family contains one more degree of freedom besides scaling: Identifying M with $T^*\mathbb{P}^1$ by a hyper-Kähler rotation, one sees that the harmonic vector fields X_c defined after Corollary 3.14 are all equal to the scaling field $X = r\partial_r$ along the fibers of $T^*\mathbb{P}^1$. The flow of X accounts for the extra parameter.

5.5. **Discussion of the gauge fixing issue.** The estimates on the rate of convergence of g to g_0 in our examples required the choice of a diffeomorphism Φ between C and M near infinity. We made a rather natural choice for Φ, but conceivably the rates we obtain could be improved upon by applying a suitable gauge fixing. In this section, we collect together some simple remarks on this issue.

(1) Consider the ordinary double point C from Section 5.4. If one smooths C by adding on both constant *and* linear terms to its defining equation and applies our theory, then one finds that the rate of convergence of the resulting AC Calabi-Yau metric to its asymptotic cone is at least $-\frac{n}{n-1}$. But every such smoothing is biholomorphic to the standard one considered in Section 5.4, so these metrics are still isometric to Stenzel, where we know that the rate is in fact at least $-2\frac{n}{n-1}$.

(2) If $n = 3$, there exists a connection (though not an entirely rigorous one) between the values of the rates and certain results in algebraic geometry: Consider a 3-dimensional *compact* Calabi-Yau variety Y with isolated singularities. It is reasonably well understood under what conditions Y can or cannot be deformed to a smooth Calabi-Yau 3-fold; see e.g. [21, 29]. On the other hand, if all the singularities of Y are locally isomorphic to Calabi-Yau cones C_i, then Y is widely expected to admit Calabi-Yau *metrics* with conical singularities modelled on the C_i [14, Definition 4.6]. Assuming the existence of such metrics, Chan [14] used gluing arguments to prove that Y is indeed smoothable if there exist AC Calabi-Yau manifolds M_i asymptotic to C_i at rate *strictly faster* than -3.

(2a) Our computations yield the critical rate -3 for AC smoothings of the ordinary double point, as was already pointed out by Chan. On the other hand, there do exist obstructions to smoothing Calabi-Yau varieties Y with ordinary double points [21]. This indicates that -3 is optimal.

(2b) In Sections 5.2–5.3, we found AC smoothings of rate -3 as well as -6 for certain complete intersection 3-fold cones. By [29, Theorem 3.8], compact Calabi-Yau varieties Y with singularities locally isomorphic to any of these cones are in fact always smoothable, for algebraic reasons.

(3) Everything we have said so far is consistent with conjecturing that, for a complete intersection Calabi-Yau cone C, the optimal rate for our AC Calabi-Yau metric on an affine smoothing M of C can be computed by realizing M as a member of the particular versal deformation of C constructed in [34, p. 24]; cf. Remarks 5.3 and 5.6. However, we have no evidence for this conjecture except that it holds true for the ordinary double point, as we will see in Section 5.6 below.

5.6. **The optimal rate for smoothing the ordinary double point.** We close by computing the leading term in the asymptotic expansion of Stenzel's Ricci-flat metric g on $\sum_{i=1}^{n+1} z_i^2 = 1$ relative to its tangent cone metric g_0 on $\sum_{i=1}^{n+1} z_i^2 = 0$. We use the same diffeomorphism $\Phi(z) = z + \frac{\bar{z}}{2|z|^2}$ as in Section 5.4 to identify Stenzel and its tangent cone away from compact subsets of each.

Proposition 5.9. *If $n > 2$, then $\Phi^* g - g_0$ is equal to the tracefree symmetric bilinear form* (5.6) *to leading order. This has rate $-2\frac{n}{n-1}$ and satisfies the Bianchi gauge condition relative to g_0.*

Remark 5.10. This result contradicts [16, Theorem 0.16]. To see why, notice that the proof given in [16] really only requires that M is AC, i.e. that the *conclusions* of [16, Theorem 0.15] hold for M. Moreover, the argument is based on analyzing the linearized Ricci-flat equations on C in a Bianchi gauge. Finally, as explained at the end of Section 5.1, the optimal rates for g and J in this example are a priori equal. See Remark 5.13 below for further details about the proof in [16].

Remark 5.11. The $n = 2$ case is Eguchi-Hanson, whose leading term can be written as $i\partial\bar{\partial} r^{-2}$ on $\mathbb{C}^2/\mathbb{Z}_2$. This tensor has rate -4 and is obviously in Bianchi gauge relative to the flat metric.

The proof of Proposition 5.9 is an explicit computation based on the following formula.

Lemma 5.12 (Stenzel [49]). *Up to scaling, the unique* $\mathrm{SO}(n+1)$*-invariant* AC *Calabi-Yau metric g on the standard smoothing of the ordinary double point in $\mathbb{C}^{n+1}$ is given by $\omega = i\partial\bar{\partial}(f \circ \tau)$, where $\tau(z) = |z|^2$ in $\mathbb{C}^{n+1}$, and $f(\tau) = h(w)$ with $\tau = \cosh w$, $(h'(w)^n)' = (\sinh w)^{n-1}$, and $h'(0) = 0$.*

One checks that $f(\tau) = C_n \tau^{1-(1/n)}(1 + k(\tau))$ with $k(\tau) \sim \tau^{-1}$ if $n = 2$, $k(\tau) \sim \tau^{-2} \log \tau$ if $n = 3$, and $k(\tau) \sim \tau^{-2}$ if $n \geq 4$. The Calabi-Yau cone metric g_0 on the ordinary double point which serves as an asymptotic model for the Stenzel metric g has Kähler potential $\tau^{1-(1/n)}$ up to scaling.

Proof of Proposition 5.9. Since $n \geq 3$, Lemma 5.12 and the above tell us that, up to scale, g is equal to $\hat{g} = (1 + O(\tau^{-2} \log \tau))\hat{g}_0$ restricted from $\mathbb{C}^{n+1}$ to the quadric, where $\hat{g}_0 = \tau^{-(1/n)} G$ and

$$G = \sum \mathrm{Re}(G_{ij})(dx_i \otimes dx_j + dy_i \otimes dy_j) + \mathrm{Im}(G_{ij})(dx_i \otimes dy_j - dy_i \otimes dx_j) \tag{5.5}$$

is the real symmetric bilinear form on $\mathbb{C}^{n+1}$ associated with the positive Hermitian matrix

$$G_{ij} = \delta_{ij} - \frac{\bar{z}_i z_j}{n\tau}.$$

We now need to pull $\hat{g}$ back by $\Phi = \mathrm{id} + \Psi$, with $\Psi(z) = \frac{\bar{z}}{2\tau}$, then subtract $\hat{g}_0$, and finally restrict to the tangent bundle of the quadric cone. Since $\tau \circ \Phi = \tau(1 + (2\tau)^{-2})$ and since we eventually only care about $O(\tau^{-1})$ relative errors, it will be enough to deal with G, treating τ as a constant.

Some preparations first: By $\mathrm{SO}(n+1)$-invariance, it suffices to work at $p_0 = (\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, 0, ..., 0)\sqrt{\tau}$ in the quadric cone. At this point, the tangent spaces to the cone and its link are given by

$$T_{\mathrm{cone}} = \ker\Big(\sum x_l dx_l - y_l dy_l\Big) \cap \ker\Big(\sum x_l dy_l + y_l dx_l\Big) = \mathrm{span}(\partial_{x_1} + \partial_{y_2}, \partial_{y_1} - \partial_{x_2}) \oplus \mathbb{R}^{2(n-1)}_{\mathrm{last}},$$

$$T_{\mathrm{link}} = \ker(d\tau) \cap T_{\mathrm{cone}} = \ker(dx_1 + dy_2) \cap T_{\mathrm{cone}} = \mathrm{span}(\partial_{y_1} - \partial_{x_2}) \oplus \mathbb{R}^{2(n-1)}_{\mathrm{last}}.$$

We can then determine $((\mathrm{id}+\Psi)^* - \mathrm{id}^*)G$ to leading order, $O(\tau^{-1})$, by computing the action of $(\mathrm{id}+\Psi)^* - \mathrm{id}^*$ on the right-hand side of (5.5) using the Leibniz rule. The first contribution, gotten by acting on the $\mathrm{Re}(G_{ij})$ and $\mathrm{Im}(G_{ij})$ coefficients, is given by

$$-\frac{1}{n\tau^2}\sum \mathrm{Re}(z_i z_j)(dx_i \otimes dx_j + dy_i \otimes dy_j).$$

At p_0, it is easy to see that the restriction of this to T_{cone} vanishes, so we can just ignore it. As for the terms gotten by letting $(\mathrm{id}+\Psi)^* - \mathrm{id}^*$ act on the differentials, we first observe that, at p_0,

$$\Psi^* dx_i = d\frac{x_i}{2\tau} = \frac{dx_i}{2\tau} - \frac{x_i d\tau}{2\tau^2} = \frac{dx_i}{2\tau} - \delta_{1i}\frac{dx_1 + dy_2}{2\tau},$$
$$\Psi^* dy_i = -d\frac{y_i}{2\tau} = -\frac{dy_i}{2\tau} + \frac{y_i d\tau}{2\tau^2} = -\frac{dy_i}{2\tau} + \delta_{2i}\frac{dx_1 + dy_2}{2\tau}.$$

With this in hand, a lengthy but completely straightforward computation shows that the remaining $O(\tau^{-1})$ contributions to $((\mathrm{id}+\Psi)^* - \mathrm{id}^*)G$ are given by

$$\frac{1}{\tau}\left(\begin{pmatrix} -\frac{1}{2n} & 0 & 0 & 0 \\ 0 & \frac{1}{2n}-1 & 0 & 0 \\ 0 & 0 & 1-\frac{1}{2n} & 0 \\ 0 & 0 & 0 & \frac{1}{2n}\end{pmatrix} \oplus \mathrm{id}_{n-1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\right)$$

in terms of the real basis $\partial_{x_1}, \partial_{y_1}, \partial_{x_2}, \partial_{y_2}, (\partial_{x_i}, \partial_{y_i})_{i=3}^{n+1}$ of $\mathbb{C}^{n+1}$.

The final step in computing $h := \Phi^* g - g_0$ modulo $o(\tau^{-1})$ is to restrict the ensuing expression for the leading term of $\Phi^*\hat{g} - \hat{g}_0$ to T_{cone} and then diagonalize with respect to $g_0 = \tau^{-(1/n)} G|_{T_{\mathrm{cone}}}$. This is not difficult if one bears in mind the fact that

$$G = \begin{pmatrix} 1-\frac{1}{2n} & 0 & 0 & -\frac{1}{2n} \\ 0 & 1-\frac{1}{2n} & \frac{1}{2n} & 0 \\ 0 & \frac{1}{2n} & 1-\frac{1}{2n} & 0 \\ -\frac{1}{2n} & 0 & 0 & 1-\frac{1}{2n}\end{pmatrix} \oplus \mathrm{id}_{2(n-1)}.$$

The end result is that, at our chosen point p_0, h takes the form

$$h = \frac{1}{\tau}\mathrm{diag}\left(\begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}, \ldots, \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\right), \;\; \tau^{1-\frac{1}{n}} = r^2, \tag{5.6}$$

where r denotes the radius function of the Ricci-flat cone metric g_0 and the matrix representation is with respect to a g_0-orthonormal basis $(\partial_r, J\partial_r), (X_3, Y_3), \ldots, (X_{n+1}, Y_{n+1})$ of T_{cone} such that the last $2n-2$ vectors form a common rescaling of the Euclidean basis $\partial_{x_3}, \partial_{y_3}, \ldots, \partial_{x_{n+1}}, \partial_{y_{n+1}}$.

To finish the proof, we must compute $\mathrm{div}_{g_0}(h)$. Since this is an $\mathrm{SO}(n+1)$-invariant 1-form on the cone, it can be written as $\alpha(r)dr + \beta(r)d^c r$. Indeed, the stabilizer of p_0 in $\mathrm{SO}(n+1)$ is the standard $\mathrm{SO}(n-1)$, so that $J\partial_r$ is the only isotropy invariant tangent vector to the link at p_0, hence the only vector at p_0 that extends to an $\mathrm{SO}(n+1)$-invariant vector field on the link. We compute that

$$\alpha(r) = (\mathrm{div}_{g_0}(h))(\partial_r) = -\sum (h(X_i, \nabla_{X_i}\partial_r) + h(Y_i, \nabla_{Y_i}\partial_r)) = -\frac{1}{r}(\mathrm{tr}\, h) = 0,$$

where all derivatives are with respect to g_0 and where we have used the fact that $h(\partial_r, X) = 0$ for all vector fields X on the cone. And similarly, we have $\beta(r) = 0$, so that $\mathrm{div}_{g_0}(h) = 0$, as desired. □

Remark 5.13. As discussed in Remark 5.10, the result of Proposition 5.9 indicates that the proof of [16, Theorem 0.16] is not correct. (There is no problem with Theorems 0.13 or 0.15 of [16].) The proof in [16] is valid if (7.106) and (7.137) hold. The Stenzel metric satisfies (7.106) but not (7.137). As a result, (7.136) and (7.138) also fail. In order to see this, one can compute directly that

$$\dot{J}(\partial_r) = \dot{J}(J\partial_r) = 0, \quad X \perp \partial_r, J\partial_r \implies \dot{J}X = \frac{1}{\tau}J\bar{X},$$

where the bar denotes complex conjugation on $\mathbb{C}^{n+1}$. Thus, $\dot{J} = \dot{J}^S$, so that $(d\dot{J}^S)^{SH} = 0$ (compare below). We also learn that (7.133) holds with $c = 2\frac{n}{n-1}$. Notice here that $J(r\partial_r) \perp \mathfrak{so}(n+1)$.

It turns out that the proof of Proposition 7.135 is correct, except for the fact that (7.134) should contain a correction term which vanishes when (7.137) holds. More precisely, the result cited from [B, p. 363] on p. 557 of [16] is only applicable whenever (7.137) holds. In general, we have that

$$\frac{1}{2}\Box^b = (\bar{\partial}^b)^*\bar{\partial}^b + \bar{\partial}^b(\bar{\partial}^b)^* + (n-3)\mathcal{J}L_{\partial_\theta},$$

where $\Box^b$ denotes the part of the Lichnerowicz Laplacian on the link that only involves horizontal covariant derivatives, as in (7.146). As a consequence, (7.151) should read $\underline{\mu} \geq -2(n-3)c$, so that (7.155) becomes a tautology. It is however still possible to prove that if (7.106) holds, then

$$(R^\circ)^b \leq \lambda \,\mathrm{id}^b \implies (n-3)c \leq \lambda.$$

With regard to (7.106), we mention that while $(d\dot{J})^{SH} = 0$ always holds, the remaining statements can fail. However, the known counterexample does in fact satisfy the conclusions of Theorem 0.16. It may even be the case that $\dot{J} = \dot{J}^S$ always holds true, except in some special situations.

Appendix A. An $i\partial\bar{\partial}$-lemma for AC Kähler manifolds

Recall that a complex manifold is 1-*convex* if it carries a smooth proper function which is strictly plurisubharmonic outside a compact subset.

Example A.1. (i) AC Kähler manifolds are 1-convex almost by definition.

(ii) Let X be a compact complex manifold with a smooth divisor D whose normal bundle $N_{D/X}$ is ample. Then $M = X \setminus D$ is 1-convex; an exhaustion function can be constructed by transplanting $\frac{1}{h}$ from the total space of $N_{D/X}$ to M, where h is a positively curved Hermitian metric on $N_{D/X}$.

The 1-convexity of a complex manifold M is equivalent to the existence of a *Remmert reduction*, that is, a proper holomorphic map $\pi : M \to V$ onto a normal Stein variety V with at worst finitely many singularities (in particular, $V \hookrightarrow \mathbb{C}^N$ and the singularities are algebraic) such that

- π has connected fibers,
- $\mathrm{Exc}(\pi)$ is the maximal positive-dimensional analytic subset of M,
- π is an isomorphism onto its image away from $\mathrm{Exc}(\pi)$, and
- $\pi^*\mathcal{O}_V = \mathcal{O}_M$.

The map $\pi : M \to V$ is in fact unique up to isomorphism, and V is then often simply referred to as the Remmert reduction of M. See Grauert [24, §2] for details and further references.

The following proposition and corollary are due to van Coevering [51, §4.2].

Proposition A.2. *Let M be a 1-convex manifold with trivial canonical bundle.*

(i) *We have $H^k(M, \mathcal{O}_M) = 0$ for each $k \geq 1$.*

(ii) *If $n = \dim M > 2$ and if K_R denotes the closed R-sublevel of a smooth proper function as in the definition of 1-convex, then $H^k(M \setminus K_R, \mathcal{O}_M) = 0$ for all $k \in \{1, ..., n-2\}$ and $R \gg 1$.*

Part (ii) is a fairly delicate result: According to [25, Chapter V, §1.4], $H^{n-d-1}(\mathbb{C}^n \setminus \mathbb{C}^d, \mathcal{O}_{\mathbb{C}^n}) \neq 0$, and $H^1(\mathbb{C}^2 \setminus 0, \mathcal{O}_{\mathbb{C}^2})$ is in fact infinite-dimensional; see also [33, Theorem 8.9.2].

Corollary A.3. *Let M be an* AC *Kähler manifold with trivial canonical bundle.*

(i) *If α is an exact real $(1,1)$-form on M, then there exists $u \in C^\infty(M)$ such that $\alpha = i\partial\bar{\partial}u$.*

(ii) *If $n = \dim M > 2$ and if α is an exact real $(1,1)$-form on $M \setminus K$ for some compact $K \subset M$, then there exist a compact $K' \supset K$ and $u \in C^\infty(M \setminus K')$ such that $\alpha = i\partial\bar{\partial}u$ on $M \setminus K'$.*

We now give the proof of Proposition A.2. Part (i) is more standard and suffices for applications to quasiprojective manifolds [18]. Part (ii) is only needed to prove Theorem 2.4 in full generality.

Proof of Proposition A.2. Part (i) follows from a result of Grauert and Riemenschneider to the effect that $H^k(M, \mathcal{O}(K_M)) = 0$ for all $k \geq 1$ on every 1-convex manifold M; see [26, §2.4, Korollar].

To prove Part (ii), let $\pi : M \to V$ denote the Remmert reduction of M as above. We can clearly replace M by V in the assertion we need to prove. Now, there exists a long exact sequence

$$H^{k-1}(V, \mathcal{O}_V) \to H^{k-1}(V \setminus K_R, \mathcal{O}_V) \to H^k_{K_R}(V, \mathcal{O}_V) \to H^k(V, \mathcal{O}_V);$$

see [3, Chapter I, §1(a)] for the definitions. Since V is Stein, this implies that

$$H^k(V \setminus K_R, \mathcal{O}_V) \cong H^{k+1}_{K_R}(V, \mathcal{O}_V)$$

for $k \geq 1$. We now wish to employ [3, Chapter I, §3, Theorem 3.1(a)] to see that $H^{k+1}_{K_R}(V, \mathcal{O}_V) = 0$ for $k \in \{0, ..., n-2\}$, which would then clearly finish the proof.

Now, on one hand, K_R can be written as the intersection of open sets with strictly pseudoconvex boundaries, which means that K_R is a Stein compact in the sense of [3, p. 29].

The other condition we need to check is that the local rings of V have depth n, or equivalently, that V is Cohen-Macaulay. Rational implies Cohen-Macaulay [35, Theorem 5.10], and an isolated singularity is rational if there exists an L^2 holomorphic volume form in a deleted neighborhood [10, Proposition 3.2]. But this is clearly the case here because K_M is trivial. $\square$

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